# (Mem)Branes and Integrable Systems

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$\longrightarrow$ Dynamics of D-Branes in $\mathcal{N}$ =4 SYM
(AA '06)
$\longrightarrow$ Worldvolume description of membranes
(Work in progress)
$\longrightarrow$ Lessons from Quantum Spin Chains in both
theories

$\mathcal{N} = 4 SYM$ on $R \times S^3$

$S = \int d^4 x tr \left[ \mathcal{F}^2 + (D\Phi)^2 + g^2 [\Phi^i, \Phi^j]^2 + Fermions \right]$

Computation of Anomalous Dimensions

Dilatation Operator $\rightarrow$ Matrix Model $\stackrel{1/N \rightarrow 0}{\rightarrow}$ Spin Chain

$D = D_0 + \lambda D_1 + \lambda^2 D_2 + \cdots$

Local Composite Operators $\rightarrow$ States of the Dilatation Operator

Traces $\rightarrow$ Closed Spin Chains

e.g $trZ^k \rightarrow | \uparrow\uparrow \cdots \uparrow >$ $tr(ZWZZ) \rightarrow | \uparrow\downarrow\uparrow\uparrow >$

$Z = \Phi^i + i\Phi^2, W = \Phi^3 + i\Phi^4$: $SU(2)$ Sector

Baryon Like operators of $O(N)$: Giant Gravitons

$\epsilon_{i_1\cdots i_N}\epsilon^{j_1\cdots j_N}(Z^{i_1}_{j_1}\cdots Z^{i_N}_{J_N})$

Maximal BPS Giant: Charged Under $U(1)$

Multi-Charge BPS Giants:

e.g. $\epsilon^{j_1\cdots j_{N-1}j_N}_{i_1\cdots i_{N-1}i_N}((Z^{i_1}_{j_1}\cdots Z^{i_{N-1}}_{J_{N-1}})(W^{i_N}_{j_N})$

$\epsilon^{j_1\cdots j_{N-1}j_N}_{i_1\cdots i_{N-1}i_N}((Z^{i_1}_{j_1}\cdots Z^{i_{N-1}}_{J_{N-1}})(\Psi^{i_N}_{j_N})$

Non-BPS Giants

$\epsilon^{j_1\cdots j_{N-1}j_N}_{i_1\cdots i_{N-1}i_N}((Z^{i_1}_{j_1}\cdots Z^{i_{N-1}}_{J_{N-1}})(WZZWZWW\cdots W)^{i_N}_{j_N}$

SYM Picture of an open string ending on a brane

Mixing of Non-BPS giants: Notions of planarity etc are much more involved that those for the traces

Non BPS Giants:
$\Theta(Z^{N-1},C)=\epsilon^{j_i\cdots j_{N-1}j_N}_{i_i\cdots i_{N-1}i_N}(Z^{i_1}_{j_1}\cdots Z^{i_{N-1}}_{j_{N-1}})(C^{i_N}_{j_N})$

$C=(WZZWZZWWW\cdots ZW)$

SYM picture of Open Strings Coupled To Giant Gravitons

$<\Theta(Z^{N-1},C),\Theta(Z^{N-1},C)>_{Free}\sim([N-1]!)^3N^{|C|+1}$

$\Theta(Z^{N-1},ZC')\sim\frac{1}{N^2}det[Z]tr[C']$

Closure Under Dilatation...

$D\Theta(Z^{N-1},C)=\sum_i\Theta(Z^{N-1},C_i)$

$$C = (WZW \cdots ZZW)$$

$$D = D_0 + \frac{\lambda}{N} D_1 + \frac{\lambda^2}{N^2} D_2 + O(\lambda^3)$$

One Loop:

$$D_1 = -tr[A^{\dagger Z}, A^{\dagger W}][A_Z, A_W]$$

Interaction of Boundary $W$ with the $Z$s on the Brane Does Not Produce Anything of $O(1)$

Mixing is Described Once Again by (Open) Spin Chains

$$D_1 = \sum_{l=1}^{L-1} (\lambda) Q_1^Z Q_J^Z (I - P_{l,l+1}) Q_1^Z Q_J^Z$$

(Berenstein and Vasquez '05)

The One Loop Bethe Ansatz:

Ground State:

$$\epsilon^{j_i \cdots j_{N-1} j_N}_{i_i \cdots i_{N-1} i_N} (Z^{i_1}_{j_1} \cdots Z^{i_{N-1}}_{j_{N-1}})(WWWW \cdots WWW)^{i_N}_{j_N}$$

2 Magnon State

$$|\Psi_2 >= \sum_{x<y} \Psi(x,y)|x,y>$$

with

$$\Psi(x,y) = \sum_p \sigma(p) A(k_i, k_2) e^{i(k_1 x_1 + k_2 x_2)}$$

$$\frac{\alpha(k_i)\beta(k_i)}{\alpha(-k_i)\beta(-k_i)} = \prod_{j \neq i} \frac{S(-k_i, k_j)}{S(k_i, k_j)}$$

$$\begin{aligned} \alpha(-k) &= e^{2ik} - e^{ik} \\ \beta(k) &= e^{i(L-1)k} - e^{iLk} \end{aligned}$$

$$S(k_1, k_2) = 1 - 2e^{ik_2} + e^{i(k_1+k_2)}$$

$$E = 4\lambda \sum_i \left( \sin^2(\frac{k_i}{2}) \right)$$

One Loop Dilatation Operator $\longrightarrow$ Heisenberg Chain with Open Boundary Conditions

Closed Chains in $su(2)$ Sector $\in$ Well Known Integrable Long Ranged Spin Chains (Till $O(\lambda^3)$)

Not So in the Open String Sector

$$C = (WZW \cdots ZZW)$$

Two Loops:

$$D_2 = tr(\frac{1}{2}[A_Z, A_W][A^{\dagger Z}, [A_Z, [A^{\dagger Z}, A^{\dagger W}]]]$$

$$+\frac{1}{2}[A_Z, A_W][A^{\dagger W}, [A_W, [A_Z, A_W]]]$$

$$+N[A^{\dagger Z}, A^{\dagger W}][A_Z, A_W])$$

(BKS 2003)

$$H = \sum_{l=1}^{J-1} -(2\lambda^2)Q_1^Z Q_J^Z (I - P_{l,l+1}) Q_1^Z Q_J^Z$$

$$+\frac{\lambda^2}{2} \sum_{l-1}^{J-2} QZ_1 Q_J^Z (I - P_{l,l+2}) Q_1^Z Q_J^Z$$

$$+Q_1^Z Q_J^Z ((I - Q_2^Z) + (I - Q_2^Z)) Q_1^Z Q_J^Z$$

Ground State

$$\epsilon_{i_i \cdots i_{N-1} i_N}^{j_i \cdots j_{N-1} j_N} (Z_{j_1}^{i_1} \cdots Z_{j_{N-1}}^{i_{N-1}})(WWWW \cdots WWW)_{j_N}^{i_N}$$

Asymptotic solutions.

Boundary scattering

$$\begin{aligned} \alpha(-k) &= (1-2g) + ge^{-ik} - ge^{ik} - gE_1(k) \\ \beta(k) &= e^{ik(L+1)}\Big((1-2g) + ge^{ik} - ge^{-ik} - gE_1(k)\Big) \end{aligned}$$

Bulk scatterring

$$\frac{\alpha(k_i)\beta(k_i)}{\alpha(-k_i)\beta(-k_i)} = \prod_{j\neq i} S(-k_i,k_j)S(k_j,k_i)$$

$$S(p,p') = \frac{\phi(p)-\phi(p')+i}{\phi(p)-\phi(p')-i}$$

$$\phi(p) = \frac{1}{2}\cot\left(\frac{p}{2}\right)\sqrt{1+8\lambda\sin^2\left(\frac{p^2}{2}\right)}$$

*Not* the solution to the two loop problem!

# Spin Chains for Bosonic Membranes

Matrix quantum mechanics with discrete spectra

$$H = \mathrm{Tr}\left(\Pi_i \Pi_i + \mu_i^2 X_i X_i + \text{Interaction Terms}\right)$$

Expamples

$\longrightarrow$Dilatation operator of $\mathcal{N} = 4$ SYM on $R \times S^3$.

$\longrightarrow$Matrix theories on plane wave backgrounds

Perturbative computation of large $N$ spectrum

$$H = \mathrm{Tr}(\mu_i A^{\dagger i} A_i + \cdots)$$

Single trace states

$$\frac{1}{N^{J/2}} \mathrm{Tr}\left(A^{\dagger i_1} \ldots A^{\dagger i_J}\right) |0> = |i_1 i_2 \cdots i_J>$$

$$\mathcal{E}_0 = n_1 \mu_1 + n_2 \mu_2 + \cdots$$

Corrections

$$\delta \mathcal{E}_0 \leftrightarrow \text{Quantum spin chains}$$

Matrix Models and Spin Chains (Rajeev and Lee '98)

$$H = \text{Tr}\left(A^{\dagger i}A_i + \frac{\lambda}{N}\Psi_{ij}^{kl}A^{\dagger i}A^{\dagger j}A_kA_l + ...\right)$$

$$[(A_i)^\alpha_\beta, (A^{\dagger j})^\mu_\nu] = \delta_i^j\delta_\nu^\alpha\delta_\beta^\mu$$

Single trace states $\leftrightarrow$ Spin chains

$$\frac{1}{N^{J/2}}\text{Tr}\left(A^{\dagger i_1}\cdots A^{\dagger i_J}\right)|0> = |i_1 i_2 \cdots i_J>$$

$$\frac{1}{N}\text{Tr}(A^{\dagger i}A^{\dagger j}A_kA_l)|i_1\cdots i_J> = \delta_k^{i_m}\delta_l^{i_{m+1}}|i_1\ldots i_{m-1}ij i_{m+2}\cdots i_J>$$

e.g

$$\frac{1}{N}\text{Tr}(A^{\dagger i}A^{\dagger j}A_iA_j) = \sum_l P_{l,l+1}, \text{Tr}(A^{\dagger i}A_i) = J$$

Bosonic membranes in $D+2$ flat background

$$S = -T \int d^3\sigma \sqrt{-det h_{\alpha\beta}}$$

The brane tension

$$T = \frac{1}{2\pi l_p^3}$$

Gauge fixed matrix regularized Hamiltonian

$$H = g^3 \mathrm{Tr}\left(\Pi_i \Pi_i\right) - \frac{1}{4g^3} \mathrm{Tr}\left([X^i, X^j][X^i, X^j]\right),$$

$i, j = 1 \cdots D$ and

$$g^3 = 2\pi l_p^3.$$

$\rightarrow$Classical flat directions e.g $[X^i, X^j] = 0$

$\rightarrow$Numerical and other analyses suggest that quantum spectrum of Bosonic membranes is discrete.

$\rightarrow$Quantum mechanical effective potential has a mass term

$\rightarrow$ Matrix models with discrete spectra $\leftrightarrow$ Quantum spin chains

Dynamical mass generation

$$S = \int dt \frac{1}{2g_{YM}^3} Tr\left(\dot{D}_i \dot{D}_i + m^2 D_i D_i - \frac{\hbar}{2}[D_i, D_j]^2\right) - \hbar \int dt \frac{m^2}{2g_{YM}^3} Tr(D_i D_i).$$

$$m^2 = m_1^2 + \hbar m_2^2 + \hbar^2 m_3^2 + \cdots$$

Propagator

$$\left\langle (D_i)^a_b(p)(D_j)^c_d(-p)\right\rangle = \frac{\delta^a_d \delta^c_b \delta_{ij}}{\frac{1}{g_{YM}^3}(p^2 + m^2 - \hbar m^2) + \Sigma(p)}$$

$$\Sigma(p) = \hbar\Sigma_1(p) + \hbar^2\Sigma_2(p) + \cdots$$

One loop gap equation

$$\frac{\hbar m_1^2}{g_{YM}^3} = \hbar\Sigma_1. \Rightarrow m_1 = ((d-1)N)^{\frac{1}{3}} g$$

$$\lambda = gN^{\frac{1}{3}} \rightarrow m = \lambda\mu, \mu = (d-1)^{\frac{1}{3}}$$

Two loops:

$$m = m_1\left(1 - \frac{\hbar}{6(d-1)}\right).$$

Canonical quantization

$$H = \mathrm{Tr}\left(\frac{1}{2}\left(\Pi_i \Pi_i\right) + \frac{m^2}{2}\mathrm{Tr} D_i D_i - \frac{g^3}{4}\left([D^i, D^j][D^i, D^j]\right)\right)$$

Introduce the matrix creation and annihilation operators

$$H' = \lambda \mathrm{Tr}\left(\mu A^{\dagger i} A_i - \frac{1}{16\mu^2 N} : [A_i + A^{\dagger i}, A_j + A^{\dagger j}]^2 :\right)$$

$\mu$ acts a dimensionless expansion parameter for perturbation theory.

Matrix models with Chern-Simons couplings: Myers Model

$$S = \int dt \mathrm{Tr} \left( \frac{1}{2} \dot{X}_i^2 - \frac{g^3}{4} [X_i, X_j]^2 - \frac{i g^{3/2} \kappa}{3} \epsilon_{ijk} X_i X_j X_k \right).$$

$\rightarrow$0 Brane QM in $IIA$ theory.

$\rightarrow$ $\kappa$= Strength of four form flux

$$\kappa^2 = \beta g^2$$

$$m^3 = 2\lambda^3 f(\beta), f(\beta) = 1 - \frac{1}{2^{5/3}} \beta + \cdots = \lambda^3 \mu^3$$

't Hooft coupling $\lambda^3 = N g^3$

Canonically quantized Hamiltonian with the dynamically generated mass

$$H = \lambda \mathrm{Tr} (\mu A^{\dagger i} A_i - \frac{1}{16\mu^2} [A_i + A^{\dagger i}][A_j + A^{\dagger j}]$$

$$- i \frac{\sqrt{\beta}}{3(2^{3/2} \mu^{3/2})} \epsilon_{ijk} (A_i + A^{\dagger i})(A_j + A^{\dagger j})(A_k + A^{\dagger k}))$$

One Loop Spin Chains

$$\delta\mathcal{E}_1 \sim < \Psi|V|\Psi >$$

$$H^1_{so(d)} = \frac{1}{8\mu^2}\sum_l \Big(2I_{l,l+1} - 4P_{l,l+1} + 2K_{l,l+1}\Big)$$

$$H^1_{CS} = \frac{1}{8\mu^2}\sum_i ((2+\theta)I_{l,l+1} + (\theta - 4)P_{l,l+1}$$
$$+(2-\theta)K_{l,l+1})$$

$$\theta = \frac{\beta}{3\mu^2}$$

These chains are generically not integrable

Exception: Bosonic membrane in D=4
QM with 2 matrices: $[D_1, D_2]^2 \rightarrow SO(2)$ invariance

$$H = \frac{1}{2}(\frac{9}{4}J - \frac{1}{4}(\sum_{l=1}^{J}[\sigma^x(l)\sigma^x(l+1) + \sigma^z(l)\sigma^z(l+1) + 3\sigma^y(l)\sigma^y(l+1)]))$$

$$\mathcal{M}^2 = \lambda\left(\frac{6}{8}J + \sum_{i=1}^{m}\left(1 + \sin^2(\frac{p_i}{2})\right)\right)$$

$$e^{ip_k J} = \prod_{(j \neq k)=1}^{m} \mathcal{S}(p_k, p_j)$$

$$\mathcal{S}(p_1, p_2) = \frac{1 + e^{i(p_1+p_2)} + 2\delta e^{ip_2}}{1 + e^{i(p_1+p_2)} + 2\delta e^{ip_1}}, \delta = -3$$

$$p_k J = 2n_k\pi + \Theta(p_1, p_2)$$

$$\mathcal{M}^2 = \lambda\left(\frac{6}{8}J + m + \sum_{i=1}^{m}\left(\frac{n_i\pi}{J}\right)^2\right)$$

Enhancement of integrability in the large $J$ limit

$\mathcal{R}$ matrix for $so(n)$ spin chains in the vector representation.

$\mathcal{R} \in End(\mathcal{V}_1 \otimes \mathcal{V}_2)$

$$\mathcal{R}(u)_{i,j} = I_{i,j} + \frac{\hbar}{u} P_{i,j} + \frac{\hbar}{\hbar g - u} K_{i,j}, g = \left(1 - \frac{n}{2}\right)$$

The transfer matrix

$$\mathcal{T}(u) = \mathcal{R}_{01}(u)\mathcal{R}_{02}(u) \cdots \mathcal{R}_{0J}(u)$$

$$[H, \mathrm{Tr}_0 \mathcal{T}] = 0$$

Most general nearest neighbor Hamiltonian is

$$H = \alpha \sum_{l=1}^{J} \left( g P_{l,l+1} - \beta I_{l,l+1} + K_{l,l+1} \right)$$

Relative coefficient of $P$ and $K$ operators is $g$.

Large $J$ limit as a classical limit

$$\mathcal{R}(u)^{i}_{\mu\nu} = \delta_{\mu\nu} I(i) + \frac{\hbar}{u} S_{\nu\mu}(i) + \frac{\hbar}{\hbar g - u} S_{\mu\nu}(i)$$

$$[S_{\mu\nu}(i), S_{\alpha\beta}(j)] = \hbar \delta_{i,j} \left( \delta_{\nu\alpha} S_{\mu\beta} - \delta_{\mu\beta} S_{\alpha\nu} \right)$$

RTT relations

$$\frac{1}{\hbar}[T_{ab}(u), T_{cd}(v)] + \frac{1}{u-v} \left( T_{ad}(u) T_{cb}(v) - T_{ad}(v) T_{cb}(u) \right)$$
$$= \frac{-1}{\hbar g - (u-v)} \left( \delta_{a,b} T_{lc}(u) T_{ld}(v) - \delta_{c,d} T_{al}(v) T_{bl}(u) \right)$$

Rescaled variables

$$\hbar = \frac{\hbar'}{J}, u', v' = \frac{u'}{J}, \frac{v'}{J}, \mathcal{S} = \frac{S}{J}$$

$$t_{ab} = \mathcal{P} \left( e^{\int \frac{1}{u'} w} \right)_{ab}$$

$$\{w_{ij}, w_{kl}\} = \delta_{il} w_{jk} - \delta_{jl} w_{ik} - \delta_{ik} w_{jl} + \delta_{kj} w_{il}$$

$$\{t(u') \overset{\otimes}{,} t(v')\} = [r(u'-v'), t(u') \otimes t(v')]$$

$$r(x-y) = \frac{1}{x-y}(P - K)$$

Hamiltonian

$$h = \alpha \int \mathrm{Tr}(\partial w \partial w)$$

$$\{h, \mathrm{Tr}t(u)\} = 0$$

$h$ from coherent state expectation value

$$\sum_l (AP_{l,l+1} + BK_{l,l+1}) \stackrel{J\to\infty}{\to} \frac{A}{J^2} \int \mathrm{Tr}(\partial w \partial w)$$

Equation of motion

$$\frac{\partial w}{\partial t} = -4\alpha \partial [w, \partial w]$$

Lax connection

$$\begin{aligned} \mathcal{A}_x &= \frac{1}{u} w \\ \mathcal{A}_t &= -\frac{4\alpha}{u}[w, \partial w] + \frac{4\alpha}{u^2} w \end{aligned}$$

Flatness

$$[\partial_t + \mathcal{A}_t, \partial_x + \mathcal{A}_x] = 0 \Leftrightarrow \mathrm{EOM}$$

Monodromy

$$t_{ab} = \mathcal{P}\left(e^{\int \frac{1}{u} w}\right)_{ab}$$

Generic Spectrum at one loop

$$\delta\mathcal{M}^2 = \frac{\lambda}{d}\left(\alpha J + \beta m + \gamma \sum_{i=1}^{m}\left(\frac{n_i \pi}{J}\right)^2 + \mathcal{O}(\frac{1}{J^3})\right)$$

Finite Temperature Computations

$$\mathcal{Z} = \int [dD_i] e^{-\int_0^1 dt \frac{1}{2} tr\left( (\mathcal{D}_0 D_i)^2 + \omega^2 D_i D_i - \frac{(\beta\lambda)^3}{Nn}[D_1, D_2]^2 \right)}$$

$\omega = \beta\lambda$ is a natural expansion parameter

$$\mathcal{Z}_0 = \int [dD_i][dD_0] e^{-\int_0^1 dt \frac{1}{2} tr\left( (\mathcal{D} D_i)^2 + \omega^2 D_i D_i \right)}$$

Static gauge

$$(D_0)_{ab} = \delta_{ab} d_a$$

Integrate out the matrices and introduce

$$\rho(\theta) = \frac{1}{N} \sum_{i=1}^{N} \delta(\theta - \beta d_i)$$

$\rho = \frac{1}{2\pi}$ minimizes the effective action

$$S = \int d\theta d\theta' \rho(\theta) \rho(\theta') \sum_{n=1}^{\infty} \frac{1 - 2e^{-n\beta\lambda}}{n} \cos n(\theta - \theta')$$

Hagedorn temperature

$$T_H = \frac{\lambda}{\ln 2}$$

Potential application to Supermembranes

$\longrightarrow$ Finite temeperature M(atrix) theory computations through dynamical mass generation and spin chains

$\longrightarrow$ Mass deformed supermembrane theories other than the BMN matrix model