The Hubbard Chain – A Paradigmatic Integrable Model

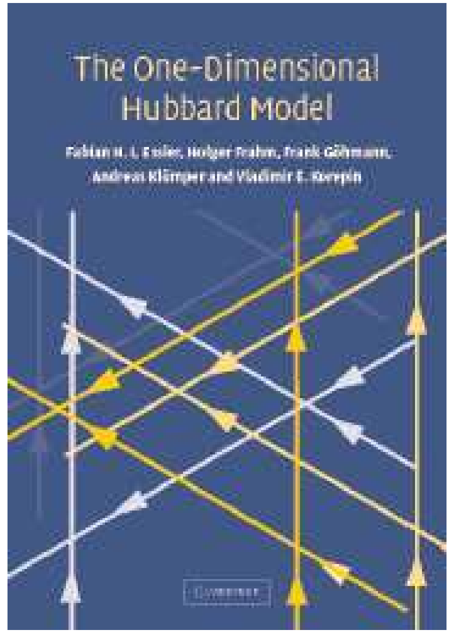
Frank Göhmann

Universität Wuppertal

Contents

Contents

- Historical remarks
- The Hubbard model
- Origin in solid state physics
- Strong coupling descendants
- Bethe ansatz solution
- Shastry's R-matrix and integrability
- Uncorrelated vacua and realization of the Yangian symmetry
- A summary and open questions



Cambridge University Press 2005
with F. H. L. Essler, H. Frahm, A. Klümper and V. E. Korepin

Historical Remarks

Physical properties

- M. C. Gutzwiller 1963, J. Hubbard 1963 (independly): Formulation (derivation) of the model in its present day form
- E. H. Lieb and F. Y. Wu 1968: Bethe ansatz solution
- M. Takahashi 1972 (74): string hypothesis, TBA equations and formulation of the thermodynamics, basis for the understanding of all elementary excitations (particle spectrum)
- F. Woynarovich 1989: finite size corrections; H. Frahm and V. E. Korepin 1990 (91): Calculation of the long distance asymptotics of correlation functions, generic case ($0 < n < 1$, h finite)
- F. H. L. Essler and V. E. Korepin 1994: Calculation of the S-matrix at for the half-filled system, clear picture of spin-charge separation
- G. Jüttner, A. Klümper and J. Suzuki 1998: Calculation of the thermodynamic properties within the quantum transfer matrix approach

Historical Remarks

Physical properties

- M. C. Gutzwiller 1963, J. Hubbard 1963 (independly): Formulation (derivation) of the model in its present day form
- E. H. Lieb and F. Y. Wu 1968: Bethe ansatz solution
- M. Takahashi 1972 (74): string hypothesis, TBA equations and formulation of the thermodynamics, basis for the understanding of all elementary excitations (particle spectrum)
- F. Woynarovich 1989: finite size corrections; H. Frahm and V. E. Korepin 1990 (91): Calculation of the long distance asymptotics of correlation functions, generic case ($0 < n < 1$, h finite)
- F. H. L. Essler and V. E. Korepin 1994: Calculation of the S-matrix at for the half-filled system, clear picture of spin-charge separation
- G. Jüttner, A. Klümper and J. Suzuki 1998: Calculation of the thermodynamic properties within the quantum transfer matrix approach

Mathematical structure:

- E. H. Lieb and F. Y. Wu 1968: Bethe ansatz solution
- B. S. Shastry 1986 (88): Construction of an R-matrix relating to the Hubbard model; analytical proof of YBE by M. Shiroishi and M. Wadati 1995
- F. H. L. Essler, V. E. Korepin and K. Schoutens 1992: SO(4) highest weight properties of Bethe ansatz states
- D. B. Uglov and V. E. Korepin 1994: Yangian symmetry; more generally (including long range) FG and V. I. Inozemtsev 1996
- M. J. Martins and P. B. Ramos 1996: Algebraic Bethe ansatz based on Shastry's R-matrix
- S. Murakami and FG 1997: Connection between Shastry's R-matrix and Yangian symmetry

The Hubbard model

- The Hamiltonian

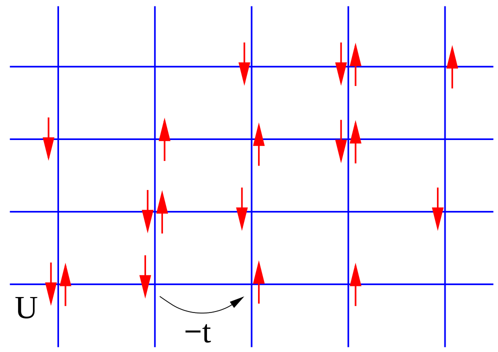
$$H = \underbrace{-t \sum_{\langle i,j \rangle} c^\dagger_{i,a} c_{j,a}}_{=:T\,, \quad \text{'hopping'}} + U \underbrace{\sum_i n_{i\uparrow} n_{i\downarrow}}_{=:D}$$

- $t \sim$ 1meV, typical energy in solids,
  $c^\dagger_{i,a}$ field operator on the lattice, creates an electron of spin a in Wannier state i

The Hubbard model

- The Hamiltonian

$$H = -t \underbrace{\sum_{\langle i,j \rangle} c^{\dagger}_{i,a} c_{j,a}}_{=:T\,,\quad \text{'hopping'}} + U \underbrace{\sum_{i} n_{i\uparrow} n_{i\downarrow}}_{=:D}$$

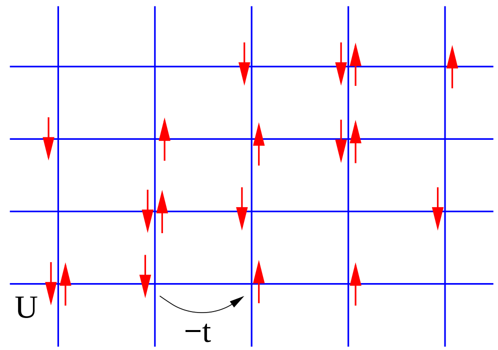


- $t \sim$ 1meV, typical energy in solids, $c^{\dagger}_{i,a}$ field operator on the lattice, creates an electron of spin a in Wannier state i

The Hubbard model

- The Hamiltonian

$$H = \underbrace{-t \sum_{\langle i,j \rangle} c^\dagger_{i,a} c_{j,a}}_{=:T\,,\quad \text{'hopping'}} + U \underbrace{\sum_i n_{i\uparrow} n_{i\downarrow}}_{=:D}$$



- $t \sim 1\text{meV}$, typical energy in solids, $c^\dagger_{i,a}$ field operator on the lattice, creates an electron of spin a in Wannier state i

Interpretation (1d)

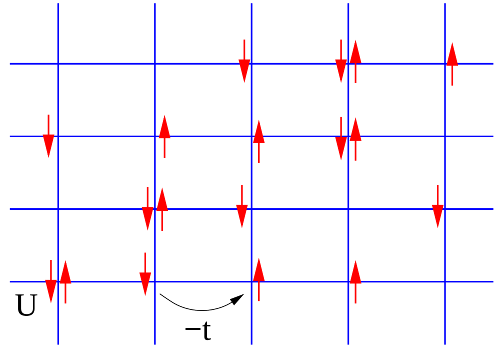
- $|\mathbf{x},\mathbf{a}\rangle = |(x_1,\ldots,x_N),(a_1,\ldots,a_N)\rangle = c^\dagger_{x_n,a_N} \ldots c^\dagger_{x_1,a_1}|0\rangle$

$$D|\mathbf{x},\mathbf{a}\rangle = \sum_{1\le m<n\le N} \delta_{x_m,x_n} |\mathbf{x},\mathbf{a}\rangle$$

D diagonal with respect to Wannier basis, counts the number of doubly occupied orbitals

- The Hamiltonian

$$H = \underbrace{-t \sum_{\langle i,j\rangle} c^\dagger_{i,a} c_{j,a}}_{=:T\,,\quad \text{'hopping'}} + U \underbrace{\sum_i n_{i\uparrow} n_{i\downarrow}}_{=:D}$$



- $t \sim 1\text{meV}$, typical energy in solids, $c^\dagger_{i,a}$ field operator on the lattice, creates an electron of spin a in Wannier state i

Interpretation (1d)

- $|\mathbf{x},\mathbf{a}\rangle = |(x_1,\ldots,x_N),(a_1,\ldots,a_N)\rangle = c^\dagger_{x_n,a_N}\ldots c^\dagger_{x_1,a_1}|0\rangle$

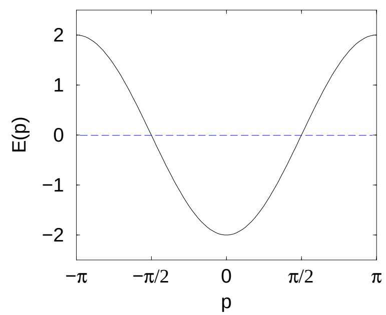
$$D|\mathbf{x},\mathbf{a}\rangle = \sum_{1\le m<n\le N} \delta_{x_m,x_n} |\mathbf{x},\mathbf{a}\rangle$$

D diagonal with respect to Wannier basis, counts the number of doubly occupied orbitals

- $U = 0 \Rightarrow H = T$ free Fermions

$$H = \sum_k -2t\cos\left(\frac{2\pi k}{L}\right)\tilde{n}_k$$

$\tilde{n}_k = \tilde{c}^\dagger_{k,a}\tilde{c}_{k,a}$ with $\tilde{c}^\dagger_{k,a}$ Fourier transform of $c^\dagger_{i,a}$

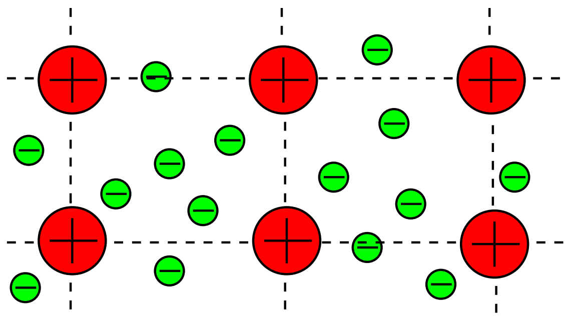


$4t$ band width, $u = U/4t$ intrinsic coupling

Origin in solid state physics

Solid a low temperature:

- positive Ions form a crystal lattice
- static lattice good starting point for studying electronic properties of solids, theoretical explanation: separation of mass scales



$$H_{el} = \sum_{i=1}^{N} \left(\frac{\mathbf{p}_i^2}{2m} + V_{Ion}(\mathbf{x}_i) \right) + \sum_{1 \le i < j \le N} V_C(\mathbf{x}_i - \mathbf{x}_j)$$

N number of electrons, $V_I(\mathbf{x})$ periodic potential of the ions

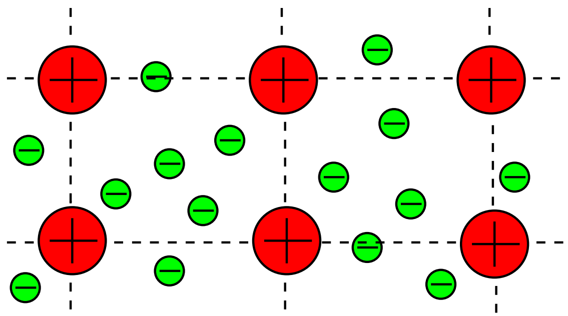
$$V_C(\mathbf{x}) = \frac{e^2}{\|\mathbf{x}\|}$$

Coulomb repulsion among the electrons

Origin in solid state physics

Solid a low temperature:

- positive Ions form a crystal lattice
- static lattice good starting point for studying electronic properties of solids, theoretical explanation: separation of mass scales



$$H_{el} = \sum_{i=1}^{N} \left(\frac{\mathbf{p}_i^2}{2m} + V_{Ion}(\mathbf{x}_i) \right) + \sum_{1 \le i < j \le N} V_C(\mathbf{x}_i - \mathbf{x}_j)$$

N number of electrons, $V_I(\mathbf{x})$ periodic potential of the ions

$$V_C(\mathbf{x}) = \frac{e^2}{\|\mathbf{x}\|}$$

Coulomb repulsion among the electrons

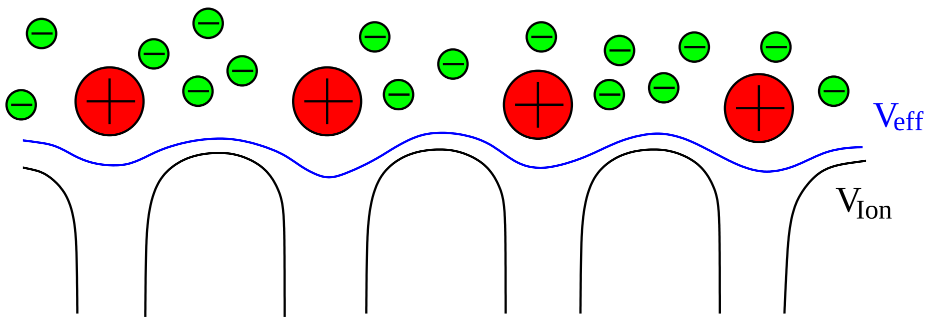
- Many body problem . . .
- Success of Solid State Physics relies on good one-body approximations to H_{el}

$$H_{el} = \sum_{i=1}^{N} \left(\frac{\mathbf{p}_i^2}{2m} + \underbrace{V_{Ion}(\mathbf{x}_i) + V_A(\mathbf{x}_i)}_{=:V_{eff}(\mathbf{x}_i)} \right) + \sum_{1 \le i < j \le N} \underbrace{\left(V_C(\mathbf{x}_i - \mathbf{x}_j) - \tfrac{1}{N-1} \left(V_A(\mathbf{x}_i) + V_A(\mathbf{x}_j) \right) \right)}_{=:U(\mathbf{x}_i, \mathbf{x}_j)}$$

- Good one-body approximations through appropriate choice of V_A: matrix elements of $U(\mathbf{x},\mathbf{y})$ between the eigenstates of the one-particle Hamiltonian $h_1(\mathbf{x},\mathbf{p}) = \frac{\mathbf{p}^2}{2m} + V_{eff}(\mathbf{x})$ must be small

Origin in solid state physics

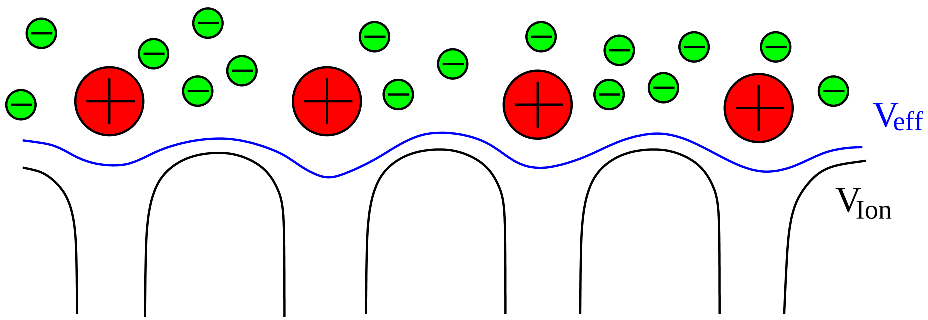
Phenomenology: Screening



- V_{Ion} screened by charge cloud $|\Psi(\mathbf{x}_1,\ldots,\mathbf{x}_N)|^2$
- Systematic approach: Density functional theory

Origin in solid state physics

Phenomenology: Screening



- Adapted bases
  (1) Bloch basis: Eigenstates of
  $h_1(\mathbf{x},\mathbf{p}) = \frac{\mathbf{p}^2}{2m} + V_{eff}(\mathbf{x})$

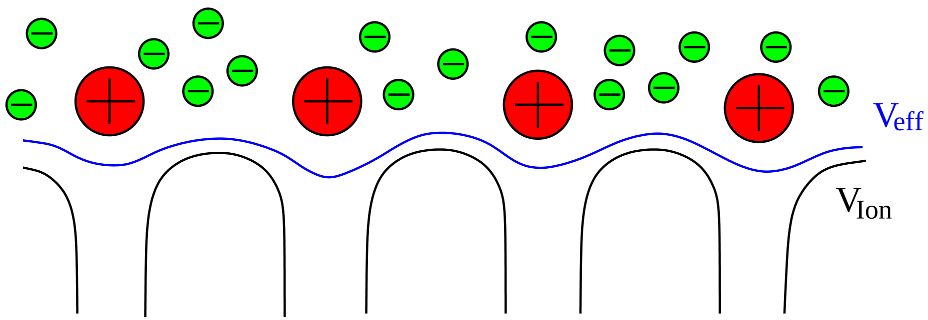
$$\varphi_{\alpha\mathbf{k}}(\mathbf{x}) = \mathrm{e}^{\mathrm{i}\langle\mathbf{k},\mathbf{x}\rangle} u_{\alpha\mathbf{k}}(\mathbf{x})$$

  $u_{\alpha\mathbf{k}}$ periodic, α band index, $\mathbf{k}$ lattice momentum,
  $c^\dagger_{\alpha\mathbf{k}}$ corresponding creation operator

- V_{Ion} screened by charge cloud $|\Psi(\mathbf{x}_1,\dots,\mathbf{x}_N)|^2$
- Systematic approach: Density functional theory

Origin in solid state physics

Phenomenology: Screening



- Adapted bases
  (1) Bloch basis: Eigenstates of
  $h_1(\mathbf{x},\mathbf{p}) = \frac{\mathbf{p}^2}{2m} + V_{eff}(\mathbf{x})$

$$\varphi_{\alpha\mathbf{k}}(\mathbf{x}) = \mathrm{e}^{\mathrm{i}\langle\mathbf{k},\mathbf{x}\rangle} u_{\alpha\mathbf{k}}(\mathbf{x})$$

  $u_{\alpha\mathbf{k}}$ periodic, α band index, $\mathbf{k}$ lattice momentum, $c^\dagger_{\alpha\mathbf{k}}$ corresponding creation operator

- V_{Ion} screened by charge cloud $|\Psi(\mathbf{x}_1,\dots,\mathbf{x}_N)|^2$
- Systematic approach: Density functional theory

(2) Wannier basis, lattice analogue of atomic wave functions

$$\phi_\alpha(\mathbf{x}-\mathbf{R}_i)$$

$$\phi_\alpha(\mathbf{x}) = \frac{1}{\sqrt{L}} \sum_{\mathbf{k}} \varphi_{\alpha\mathbf{k}}(\mathbf{x})$$

$i = 1,\dots,L =$ number of ions, $\mathbf{R}_i$ lattice vector, $c^\dagger_{\alpha i}$ corresponding creation operator

(3) Connection

$$c^\dagger_{\alpha i} = \frac{1}{\sqrt{L}} \sum_{\mathbf{k}} \mathrm{e}^{-\mathrm{i}\langle\mathbf{k},\mathbf{R}_i\rangle} c^\dagger_{\alpha\mathbf{k}}$$

Origin in solid state physics

H_{el} in Wannier representation

$$H = \sum_{\alpha,i,j,a} t_{ij}^{\alpha}\, c_{\alpha i,a}^{\dagger}\, c_{\alpha j,a} + \frac{1}{2} \sum_{\substack{\alpha,\beta,\gamma,\delta \\ i,j,k,l}} \sum_{a,b} U_{ijkl}^{\alpha\beta\gamma\delta}\, c_{\alpha i,a}^{\dagger}\, c_{\beta j,b}^{\dagger}\, c_{\gamma k,b}\, c_{\delta l,a}$$

hopping matrix elements t_{ij}^{α}

$$t_{ij}^{\alpha} = \int \mathrm{d}x^3\, \phi_{\alpha}^{*}(\mathbf{x}-\mathbf{R}_i)\,(h_1\phi_{\alpha})(\mathbf{x}-\mathbf{R}_j)$$

interaction parameters $U_{ijkl}^{\alpha\beta\gamma\delta}$

$$U_{ijkl}^{\alpha\beta\gamma\delta} = \int \mathrm{d}x^3 \mathrm{d}y^3\, \phi_{\alpha}^{*}(\mathbf{x}-\mathbf{R}_i)\phi_{\beta}^{*}(\mathbf{y}-\mathbf{R}_j)U(\mathbf{x},\mathbf{y})\phi_{\gamma}(\mathbf{y}-\mathbf{R}_k)\phi_{\delta}(\mathbf{x}-\mathbf{R}_l)$$

Origin in solid state physics

H_{el} in Wannier representation

$$H = \sum_{\alpha,i,j,a} t_{ij}^{\alpha} \, c_{\alpha i,a}^{\dagger} \, c_{\alpha j,a} + \frac{1}{2} \sum_{\substack{\alpha,\beta,\gamma,\delta \\ i,j,k,l}} \sum_{a,b} U_{ijkl}^{\alpha\beta\gamma\delta} \, c_{\alpha i,a}^{\dagger} \, c_{\beta j,b}^{\dagger} \, c_{\gamma k,b} \, c_{\delta l,a}$$

hopping matrix elements t_{ij}^{α}

$$t_{ij}^{\alpha} = \int \mathrm{d}x^3 \, \phi_{\alpha}^{*}(\mathbf{x}-\mathbf{R}_i) \, (h_1\phi_{\alpha})(\mathbf{x}-\mathbf{R}_j)$$

interaction parameters $U_{ijkl}^{\alpha\beta\gamma\delta}$

$$U_{ijkl}^{\alpha\beta\gamma\delta} = \int \mathrm{d}x^3 \mathrm{d}y^3 \, \phi_{\alpha}^{*}(\mathbf{x}-\mathbf{R}_i)\phi_{\beta}^{*}(\mathbf{y}-\mathbf{R}_j)U(\mathbf{x},\mathbf{y})\phi_{\gamma}(\mathbf{y}-\mathbf{R}_k)\phi_{\delta}(\mathbf{x}-\mathbf{R}_l)$$

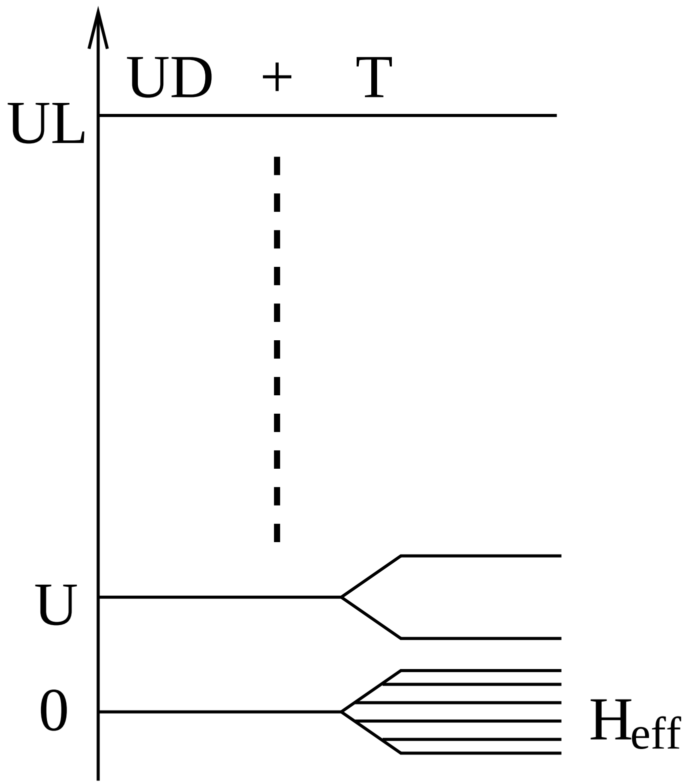
- So far H_{el} only rewritten, no approximation
- Optimal choice of the Wannier functions (optimal choice of V_A) minimises the interaction parameters
- $U_{ijkl}^{\alpha\beta\gamma\delta}$ negligible $\Rightarrow$ band model, t_{ij}^{α} band structure

- Fermi surface inside a single band $\Rightarrow$ neglect inter-band interaction, $t_{ij}^{\alpha} \longrightarrow t_{ij}$, $U_{ijkl}^{\alpha\beta\gamma\delta} \longrightarrow U_{ijkl}$, one-band model
- Usually intra-atomic interaction U_{iiii} dominant $\Rightarrow U_{ijkl} \longrightarrow U$, Hubbard!
- Applications:
  – electronic properties of solids with narrow bands
  – band magnetism of iron, cobalt, nickel
  – Mott metal-insulator transition

Strong coupling descendants

Strong coupling $U \gg |t_{ij}|$ in

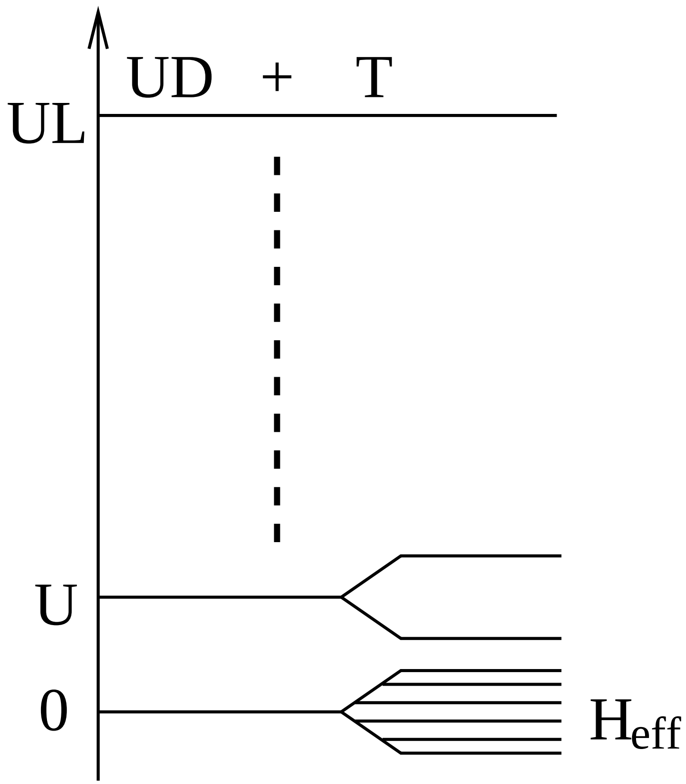
$$H = \sum_{\langle i,j \rangle} t_{ij} c^\dagger_{i,a} c_{j,a} + U \sum_i n_{i\uparrow} n_{i\downarrow}$$



Strong coupling descendants

Strong coupling $U \gg |t_{ij}|$ in

$$H = \sum_{\langle i,j\rangle} t_{ij} c^\dagger_{i,a} c_{j,a} + U \sum_i n_{i\uparrow} n_{i\downarrow}$$



- Second order (projected) degenerate perturbation theory
- $N < L$, t-J model

$$H_{t-J} = \sum_{\substack{j,k=1\\ j\neq k}}^{L} t_{jk} c^\dagger_{j,a} c_{k,a} (1-n_j) + \sum_{\substack{j,k=1\\ j\neq k}}^{L} \frac{2|t_{jk}|^2}{U} \left(S^\alpha_j S^\alpha_k - \frac{n_j n_k}{4} \right)$$

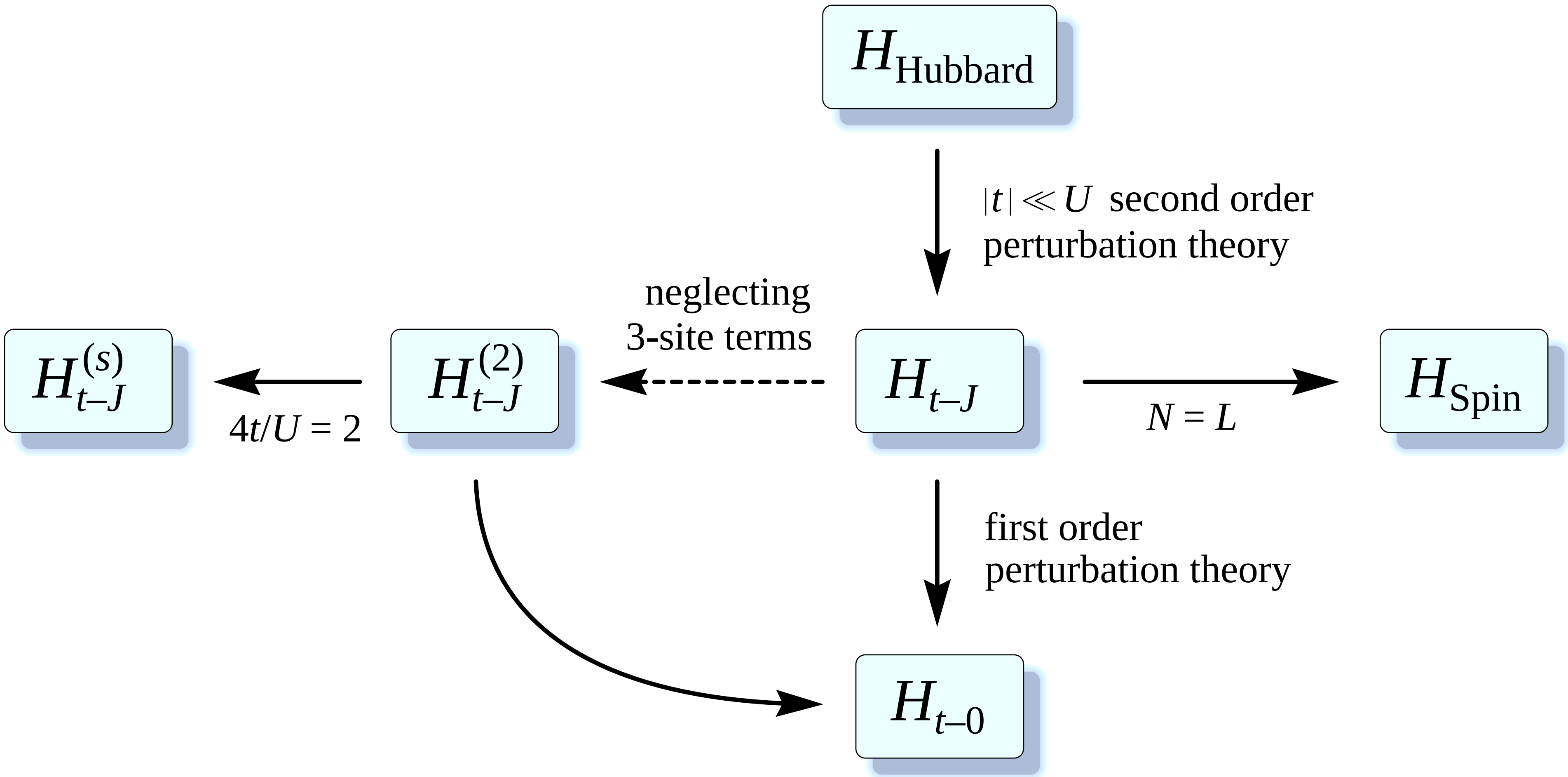
$$+ \frac{1}{U} \sum_{\substack{j,k,l=1\\ j\neq k\neq l\neq j}}^{L} t_{jk} t_{kl} \left(c^\dagger_{j,a} \sigma^\alpha_{ab} c_{l,b} S^\alpha_k - \tfrac{1}{2} c^\dagger_{j,a} c_{l,a} n_k \right) (1-n_j)$$

where $2S^\alpha_j = c^\dagger_{j,a} \sigma^\alpha_{ab} c_{j,b}$, spin operator

- $N = L$ (half-filling), Heisenberg model, Mott transition (electro-magnetic field couples like $t_{jk} \to t_{jk} e^{i\lambda_{jk}}$)

$$H_{Spin} = \sum_{\substack{j,k=1\\ j\neq k}}^{L} \frac{2|t_{jk}|^2}{U} \left(S^\alpha_j S^\alpha_k - \frac{1}{4} \right)$$

$U > 0 \Rightarrow$ exchange positive, antiferromagnetism



The various models related to the strong coupling limit of the Hubbard model

Strong coupling descendants

Strong coupling perturbation theory beyond second order

- Has appeared in a recent attempt to identify the dilatation operator of $\mathcal{N} = 4$ gauge theory in the su(2) sector (A. Rej, D. Serban and M. Staudacher 2006)

Strong coupling descendants

Strong coupling perturbation theory beyond second order

- Has appeared in a recent attempt to identify the dilatation operator of $\mathcal{N} = 4$ gauge theory in the su(2) sector (A. Rej, D. Serban and M. Staudacher 2006)
- In solid state physics vast literature on strong coupling limit (e.g. Takkahashi 1977, McDonald, Girvin and Yoshioka 1988, 90). Reasons: (1) Although Hubbbard is completely regularized it is still a truly interacting many body problem, simplifications, in particular for $d = 2, 3$, are highly appreciated. (2) Relevant for applications to real materials.
- 1d case mostly considered for validity tests

Strong coupling descendants

Strong coupling perturbation theory beyond second order

- Has appeared in a recent attempt to identify the dilatation operator of $\mathcal{N} = 4$ gauge theory in the su(2) sector (A. Rej, D. Serban and M. Staudacher 2006)
- In solid state physics vast literature on strong coupling limit (e.g. Takkahashi 1977, McDonald, Girvin and Yoshioka 1988, 90). Reasons: (1) Although Hubbbard is completely regularized it is still a truly interacting many body problem, simplifications, in particular for $d = 2, 3$, are highly appreciated. (2) Relevant for applications to real materials.
- 1d case mostly considered for validity tests

- Application in 1d (Takahashi 77)

  Up to the order t^4/U^3 the ground state energy of the Hubbard chain at half-filling can be expressed in terms of ground state correlation functions of the Heisenberg chain

$$E = \frac{t^2}{U} \sum_j \big(\langle \sigma_j^\alpha \sigma_{j+1}^\alpha \rangle_0 - 1 \big)$$

$$\frac{t^4}{U^3} \sum_j \big(4(1 - \langle \sigma_j^\alpha \sigma_{j+1}^\alpha \rangle_0) + \langle \sigma_j^\alpha \sigma_{j+2}^\alpha \rangle_0 - 1 \big)$$

On the other hand, the ground state energy of the half-filled Hubbard model has a large U expansion (convergent for $U > 4t$, Takahashi 1971). Comparison yields

$$\langle \sigma_j^z \sigma_{j+2}^z \rangle_0 = \frac{1}{3} - \frac{16 \ln 2}{3} + 3\zeta(3)$$

for next-to-nearest neighbour zz-correlator.

Bethe ansatz solution

Bethe ansatz eigenstates for N electrons and M down are characterised by two types of quantum numbers $\mathbf{k} = (k_1, \dots, k_N)$ and $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_M)$

$$|\psi_{\mathbf{k},\boldsymbol{\lambda}}\rangle = \frac{1}{N!} \sum_{x_1,\dots,x_N=1}^{L} \sum_{a_1,\dots,a_N=\uparrow,\downarrow} \psi(\mathbf{x};\mathbf{a}|\mathbf{k};\boldsymbol{\lambda})|\mathbf{x},\mathbf{a}\rangle\,,$$

where $\psi(\mathbf{x};\mathbf{a}|\mathbf{k};\boldsymbol{\lambda})$ is the N-particle Bethe ansatz wave function. It depends on the relative ordering of the coordinates x_j. To any ordering a $Q \in \mathfrak{S}^N$ can be assigned,

$$1 \le x_{Q(1)} \le x_{Q(2)} \le \cdots \le x_{Q(N)} \le L$$

This divides the configuration space of N electrons into $N!$ sectors labeled by the permutations Q. In sector Q

$$\psi(\mathbf{x};\mathbf{a}|\mathbf{k};\boldsymbol{\lambda}) = \sum_{P\in\mathfrak{S}^N} \operatorname{sign}(PQ)\langle \mathbf{a}Q|\mathbf{k}P,\boldsymbol{\lambda}\rangle\, \mathrm{e}^{\mathrm{i}\langle \mathbf{k}P,\mathbf{x}Q\rangle}$$

with spin dependent amplitudes $\langle \mathbf{a}Q|\mathbf{k}P,\boldsymbol{\lambda}\rangle$

Bethe ansatz solution

Bethe ansatz eigenstates for N electrons and M down are characterised by two types of quantum numbers $\mathbf{k} = (k_1, \dots, k_N)$ and $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_M)$

$$|\psi_{\mathbf{k},\boldsymbol{\lambda}}\rangle = \frac{1}{N!} \sum_{x_1,\dots,x_N=1}^{L} \sum_{a_1,\dots,a_N=\uparrow,\downarrow} \psi(\mathbf{x};\mathbf{a}|\mathbf{k};\boldsymbol{\lambda})|\mathbf{x},\mathbf{a}\rangle\,,$$

where $\psi(\mathbf{x};\mathbf{a}|\mathbf{k};\boldsymbol{\lambda})$ is the N-particle Bethe ansatz wave function. It depends on the relative ordering of the coordinates x_j. To any ordering a $Q \in \mathfrak{S}^N$ can be assigned,

$$1 \le x_{Q(1)} \le x_{Q(2)} \le \cdots \le x_{Q(N)} \le L$$

This divides the configuration space of N electrons into $N!$ sectors labeled by the permutations Q. In sector Q

$$\psi(\mathbf{x};\mathbf{a}|\mathbf{k};\boldsymbol{\lambda}) = \sum_{P\in\mathfrak{S}^N} \operatorname{sign}(PQ)\langle \mathbf{a}Q|\mathbf{k}P,\boldsymbol{\lambda}\rangle\, \mathrm{e}^{\mathrm{i}\langle \mathbf{k}P,\mathbf{x}Q\rangle}$$

with spin dependent amplitudes $\langle \mathbf{a}Q|\mathbf{k}P,\boldsymbol{\lambda}\rangle$

The 'charge momenta' k_j, $j = 1, \dots, N$, and λ_ℓ, and 'spin rapidities' $\ell = 1, \dots, M$, are complex numbers that satisfy the Lieb-Wu equations

$$\mathrm{e}^{\mathrm{i}k_j L} = \prod_{\ell=1}^{M} \frac{\lambda_\ell - \sin k_j - \mathrm{i}u}{\lambda_\ell - \sin k_j + \mathrm{i}u}\,, \quad j = 1, \dots, N$$

$$\prod_{j=1}^{N} \frac{\lambda_\ell - \sin k_j - \mathrm{i}u}{\lambda_\ell - \sin k_j + \mathrm{i}u} = \prod_{\substack{m=1\\ m\neq\ell}}^{M} \frac{\lambda_\ell - \lambda_m - 2\mathrm{i}u}{\lambda_\ell - \lambda_m + 2\mathrm{i}u}$$

$$\ell = 1, \dots, M$$

Bethe ansatz solution

Bethe ansatz eigenstates for N electrons and M down are characterised by two types of quantum numbers $\mathbf{k} = (k_1, \ldots, k_N)$ and $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_M)$

$$|\psi_{\mathbf{k},\boldsymbol{\lambda}}\rangle = \frac{1}{N!} \sum_{x_1,\ldots,x_N=1}^{L} \sum_{a_1,\ldots,a_N=\uparrow,\downarrow} \psi(\mathbf{x};\mathbf{a}|\mathbf{k};\boldsymbol{\lambda}) |\mathbf{x},\mathbf{a}\rangle \, ,$$

where $\psi(\mathbf{x};\mathbf{a}|\mathbf{k};\boldsymbol{\lambda})$ is the N-particle Bethe ansatz wave function. It depends on the relative ordering of the coordinates x_j. To any ordering a $Q \in \mathfrak{S}^N$ can be assigned,

$$1 \le x_{Q(1)} \le x_{Q(2)} \le \cdots \le x_{Q(N)} \le L$$

This divides the configuration space of N electrons into $N!$ sectors labeled by the permutations Q. In sector Q

$$\psi(\mathbf{x};\mathbf{a}|\mathbf{k};\boldsymbol{\lambda}) = \sum_{P \in \mathfrak{S}^N} \mathrm{sign}(PQ) \langle \mathbf{a}Q|\mathbf{k}P,\boldsymbol{\lambda}\rangle \, \mathrm{e}^{\mathrm{i}\langle \mathbf{k}P,\mathbf{x}Q\rangle}$$

with spin dependent amplitudes $\langle \mathbf{a}Q|\mathbf{k}P,\boldsymbol{\lambda}\rangle$

The 'charge momenta' k_j, $j = 1, \ldots, N$, and λ_ℓ, and 'spin rapidities' $\ell = 1, \ldots, M$, are complex numbers that satisfy the Lieb-Wu equations

$$\mathrm{e}^{\mathrm{i}k_j L} = \prod_{\ell=1}^{M} \frac{\lambda_\ell - \sin k_j - \mathrm{i}u}{\lambda_\ell - \sin k_j + \mathrm{i}u} \, , \quad j = 1, \ldots, N$$

$$\prod_{j=1}^{N} \frac{\lambda_\ell - \sin k_j - \mathrm{i}u}{\lambda_\ell - \sin k_j + \mathrm{i}u} = \prod_{\substack{m=1 \\ m \ne \ell}}^{M} \frac{\lambda_\ell - \lambda_m - 2\mathrm{i}u}{\lambda_\ell - \lambda_m + 2\mathrm{i}u}$$

$$\ell = 1, \ldots, M$$

The Bethe eigenstates are joint eigenstates of the Hubbard Hamiltonian and the momentum operator with eigenvalues

$$E = -2 \sum_{j=1}^{N} \cos k_j + u(L - 2N)$$

$$P = \left[\sum_{j=1}^{N} k_j \right] \mod 2\pi$$

Bethe ansatz solution

The Bethe ansatz equations together with the expressions for energy and momentum can be used to obtain

- The ground state properties (ground state energy, density and magnetisation, spin and charge susceptibilities) Lieb & Wu 68, Takahashi 69, 71
- TBA description of the thermodynamics, Takahashi 72, 74
- Complete picture of the elementary excitations, many authors from 70, the book
- S-matrix of elementary excitations, Essler and Korepin 94, Murakami and FG 97
- Asymptotic finite size behaviour and large-time, long-distance asymptotics of correlation functions, Woynarovich 89, Frahm and Korepin 90, 91

Bethe ansatz solution

The Bethe ansatz equations together with the expressions for energy and momentum can be used to obtain

- The ground state properties (ground state energy, density and magnetisation, spin and charge susceptibilities) Lieb & Wu 68, Takahashi 69, 71
- TBA description of the thermodynamics, Takahashi 72, 74
- Complete picture of the elementary excitations, many authors from 70, the book
- S-matrix of elementary excitations, Essler and Korepin 94, Murakami and FG 97
- Asymptotic finite size behaviour and large-time, long-distance asymptotics of correlation functions, Woynarovich 89, Frahm and Korepin 90, 91

Information that has been drawn from the wave functions (not so much)

- su(2)⊕su(2) highest weight properties

$$
\begin{aligned}
S^+|\psi_{\mathbf{k},\boldsymbol{\lambda}}\rangle &= 0 \\
S^z|\psi_{\mathbf{k},\boldsymbol{\lambda}}\rangle &= \tfrac{1}{2}(N-2M)|\psi_{\mathbf{k},\boldsymbol{\lambda}}\rangle \\
\eta^-|\psi_{\mathbf{k},\boldsymbol{\lambda}}\rangle &= 0\,, \\
\eta^z|\psi_{\mathbf{k},\boldsymbol{\lambda}}\rangle &= \tfrac{1}{2}(N-L)|\psi_{\mathbf{k},\boldsymbol{\lambda}}\rangle
\end{aligned}
$$

  counting of states (Essler, Korepin, Schoutens 92)
- norm formula conjectured (FG and Korepin 99)

Obtaining local information from the Bethe ansatz solution is, in general, non-trivial (see Maillet's talk) and seems to require algebraic techniques

Algebraic approach

The 'quantum inverse scattering method' deals with systems which are based on an associative quadratic algebra $\mathcal{T}_R$ defined in terms of its generators $T^\alpha_\beta(\lambda)$, $\alpha,\beta = 1,\ldots,d$; $\lambda \in \mathbb{C}$, by the relation

$$R(\lambda,\mu)T_1(\lambda)T_2(\mu) = T_2(\mu)T_1(\lambda)R(\lambda,\mu)$$

Here

$$T(\lambda) = \begin{pmatrix} T^1_1(\lambda) & \ldots & T^1_d(\lambda) \\ \vdots & & \vdots \\ T^d_1(\lambda) & \ldots & T^d_d(\lambda) \end{pmatrix}$$

$$T_1(\lambda) = T(\lambda) \otimes I_d$$

$$T_2(\lambda) = I_d \otimes T(\lambda)$$

where I_d is the $d \times d$ unit matrix. $R(\lambda,\mu) \in \mathrm{End}(\mathbb{C}^d \otimes \mathbb{C}^d)$ is a numerical $d^2 \times d^2$ matrix, the R-matrix, which fixes the structure of the quadratic algebra $\mathcal{T}_R$ in a similarly to the tensor of structure constants in the Lie algebra case.

Algebraic approach

The 'quantum inverse scattering method' deals with systems which are based on an associative quadratic algebra $\mathcal{T}_R$ defined in terms of its generators $T^\alpha_\beta(\lambda)$, $\alpha,\beta = 1,\dots,d$; $\lambda \in \mathbb{C}$, by the relation

$$R(\lambda,\mu)T_1(\lambda)T_2(\mu) = T_2(\mu)T_1(\lambda)R(\lambda,\mu)$$

Here

$$T(\lambda) = \begin{pmatrix} T^1_1(\lambda) & \dots & T^1_d(\lambda) \\ \vdots & & \vdots \\ T^d_1(\lambda) & \dots & T^d_d(\lambda) \end{pmatrix}$$

$$T_1(\lambda) = T(\lambda) \otimes I_d$$

$$T_2(\lambda) = I_d \otimes T(\lambda)$$

where I_d is the $d \times d$ unit matrix. $R(\lambda,\mu) \in \mathrm{End}(\mathbb{C}^d \otimes \mathbb{C}^d)$ is a numerical $d^2 \times d^2$ matrix, the R-matrix, which fixes the structure of the quadratic algebra $\mathcal{T}_R$ in a similarly to the tensor of structure constants in the Lie algebra case.

The algebra $\mathcal{T}_R$ has a rich commutative subalgebra. With the definition

$$t(\lambda) = T^\gamma_\gamma(\lambda) = \mathrm{tr}(T(\lambda))$$

we have the important result

$$[t(\lambda), t(\mu)] = 0$$

It means that $t(\lambda)$ is a generating function of a commutative subalgebra of $\mathcal{T}_R$, e.g., if $t(\lambda) = I_0 + \lambda I_1 + \lambda^2 I_2 + \dots$, then $[I_j, I_k] = 0$.
For a representation of $\mathcal{T}_R$ on the space of states of some physical system $t(\lambda)$ generates a set of mutually commuting operators which by construction are embedded into the quadratic algebra $\mathcal{T}_R$. On the one hand we may meet the requirements of Liouville's theorem in the classical limit, on the other hand, the quadratic relations of the algebra $\mathcal{T}_R$ may provide means to simultaneously diagonalize the quantum integrals of motion, generated by $t(\lambda)$.

Algebraic approach

The Yang-Baxter algebra $\mathcal{T}_R$ is associative if the R-matrix satisfies the Yang-Baxter equation

$$R_{12}(\lambda,\mu)R_{13}(\lambda,\nu)R_{23}(\mu,\nu) = R_{23}(\mu,\nu)R_{13}(\lambda,\nu)R_{12}(\lambda,\mu)$$

Under appropriate additional conditions this guarantees the existence of an infinite family of representations connected to local Hamiltonians, the ‘fundamental models’.

Algebraic approach

The Yang-Baxter algebra $\mathcal{T}_R$ is associative if the R-matrix satisfies the Yang-Baxter equation

$$R_{12}(\lambda,\mu)R_{13}(\lambda,\nu)R_{23}(\mu,\nu) = R_{23}(\mu,\nu)R_{13}(\lambda,\nu)R_{12}(\lambda,\mu)$$

Under appropriate additional conditions this guarantees the existence of an infinite family of representations connected to local Hamiltonians, the 'fundamental models'.
$\{e_\alpha^\beta\}$ gl(d) standard basis,

$$e_{j\,\alpha}^{\;\beta} = I_d^{\otimes(j-1)} \otimes e_\alpha^\beta \otimes I_d^{\otimes(L-j)}$$

Then $R_{jk}(\lambda,\mu) = R_{\beta\delta}^{\alpha\gamma} e_{j\,\alpha}^{\;\beta} e_{k\,\gamma}^{\;\delta}$ and

$$L_{j\,\beta}^{\;\alpha}(\lambda,\mu) = R_{\beta\delta}^{\alpha\gamma}(\lambda,\mu) e_{j\,\gamma}^{\;\delta}$$

defines the so-called L-matrix whose elements are operators in $\big(\mathrm{End}(\mathbb{C}^d)\big)$. The monodromy matrix

$$T(\lambda) = L_L(\lambda,\nu_L)\dots L_1(\lambda,\nu_1)$$

generates a representation of the Yang-Baxter algebra.

Algebraic approach

The Yang-Baxter algebra $\mathcal{T}_R$ is associative if the R-matrix satisfies the Yang-Baxter equation

$$R_{12}(\lambda,\mu)R_{13}(\lambda,\nu)R_{23}(\mu,\nu)=R_{23}(\mu,\nu)R_{13}(\lambda,\nu)R_{12}(\lambda,\mu)$$

Under appropriate additional conditions this guarantees the existence of an infinite family of representations connected to local Hamiltonians, the 'fundamental models'.
$\{e_\alpha^\beta\}$ gl(d) standard basis,

$$e_{j}{}_\alpha^\beta = I_d^{\otimes(j-1)}\otimes e_\alpha^\beta \otimes I_d^{\otimes(L-j)}$$

Then $R_{jk}(\lambda,\mu)=R_{\beta\delta}^{\alpha\gamma} e_{j}{}_\alpha^\beta e_{k}{}_\gamma^\delta$ and

$$L_{j}{}_\beta^\alpha(\lambda,\mu)=R_{\beta\delta}^{\alpha\gamma}(\lambda,\mu)e_{j}{}_\gamma^\delta$$

defines the so-called L-matrix whose elements are operators in $\left(\mathrm{End}(\mathbb{C}^d)\right)$. The monodromy matrix

$$T(\lambda)=L_L(\lambda,\nu_L)\dots L_1(\lambda,\nu_1)$$

generates a representation of the Yang-Baxter algebra.

If $R(\lambda_0,\nu_0)=P$ the transposition matrix, then $T(\lambda)$ defines a fundamental model: If $\nu_j=\nu_0, j=1,\dots,L$, the function $\tau(\lambda)=\ln(\hat{U}^{-1}t(\lambda))$, where U is the shift operator, generates a sequence of local, mutually commuting operators.

$$\tau'(\lambda_0)=\sum_{j=1}^{L}\underbrace{\partial_\lambda \check{R}_{j-1,j}(\lambda,\nu_0)\Big|_{\lambda=\lambda_0}}_{=:H_{j-1,j}}$$

$$\tau''(\lambda_0)=\sum_{j=1}^{L}\Big\{\partial_\lambda^2\check{R}_{j-1,j}(\lambda,\nu_0)\Big|_{\lambda=\lambda_0} - H_{j-1,j}^2-[H_{j-1,j},H_{j,j+1}]\Big\}$$

If R is of difference form $\check{R}(\lambda,\mu)=\check{R}(\lambda-\mu)$ then $\tau''(\lambda_0)=-\sum_{j=1}^{L}[H_{j-1,j},H_{j,j+1}]$. This provides an 'integrability test'.

Algebraic approach

The Yang-Baxter algebra $\mathcal{T}_R$ is associative if the R-matrix satisfies the Yang-Baxter equation

$$R_{12}(\lambda,\mu)R_{13}(\lambda,\nu)R_{23}(\mu,\nu) = R_{23}(\mu,\nu)R_{13}(\lambda,\nu)R_{12}(\lambda,\mu)$$

Under appropriate additional conditions this guarantees the existence of an infinite family of representations connected to local Hamiltonians, the 'fundamental models'.
$\{e_\alpha^\beta\}$ gl(d) standard basis,

$$e_{j\alpha}^{\ \beta} = I_d^{\otimes(j-1)} \otimes e_\alpha^\beta \otimes I_d^{\otimes(L-j)}$$

Then $R_{jk}(\lambda,\mu) = R_{\beta\delta}^{\alpha\gamma} e_{j\alpha}^{\ \beta} e_{k\gamma}^{\ \delta}$ and

$$L_{j\beta}^{\ \alpha}(\lambda,\mu) = R_{\beta\delta}^{\alpha\gamma}(\lambda,\mu) e_{j\gamma}^{\ \delta}$$

defines the so-called L-matrix whose elements are operators in $\left(\mathrm{End}(\mathbb{C}^d)\right)$. The monodromy matrix

$$T(\lambda) = L_L(\lambda,\nu_L)\dots L_1(\lambda,\nu_1)$$

generates a representation of the Yang-Baxter algebra.

If $R(\lambda_0,\nu_0) = P$ the transposition matrix, then $T(\lambda)$ defines a fundamental model: If $\nu_j = \nu_0, j = 1,\dots,L$, the function $\tau(\lambda) = \ln(\hat{U}^{-1}t(\lambda))$, where U is the shift operator, generates a sequence of local, mutually commuting operators.

$$\tau'(\lambda_0) = \sum_{j=1}^{L} \underbrace{\partial_\lambda \check{R}_{j-1,j}(\lambda,\nu_0)\Big|_{\lambda=\lambda_0}}_{=:H_{j-1,j}}$$

$$\tau''(\lambda_0) = \sum_{j=1}^{L} \Big\{ \partial_\lambda^2 \check{R}_{j-1,j}(\lambda,\nu_0)\Big|_{\lambda=\lambda_0} - H_{j-1,j}^2 - [H_{j-1,j}, H_{j,j+1}] \Big\}$$

If R is of difference form $\check{R}(\lambda,\mu) = \check{R}(\lambda-\mu)$ then $\tau''(\lambda_0) = -\sum_{j=1}^{L}[H_{j-1,j},H_{j,j+1}]$. This provides an 'integrability test'.

Hubbard model fails to pass this test! Either no difference form or non-fundamental.

Algebraic approach

$$
\check{R}(\lambda,\mu) = \frac{1}{\rho_4}\cdot
\left(\begin{array}{cccc|cccc|cccc|cccc}
\rho_1 & & & & & & & & & & & & & & & \\
 & 1 & & & -\rho_9 & & & & & & & & & & & \\
 & & 1 & & & & & & -\rho_9 & & & & & & & \\
 & & & \rho_3 & & & -\rho_6 & & & & \rho_6 & & -\rho_8 & & & \\
\hline
 & -\rho_{10} & & & 1 & & & & & & & & & & & \\
 & & & & & \rho_4 & & & & & & & & & & \\
 & & & -\rho_6 & & & \rho_5 & & & & \rho_7 & & -\rho_6 & & & \\
 & & & & & & & 1 & & & & & & \rho_{10} & & \\
\hline
 & & -\rho_{10} & & & & & & 1 & & & & & & & \\
 & & & \rho_6 & & & \rho_7 & & & \rho_5 & & & \rho_6 & & & \\
 & & & & & & & & & & \rho_4 & & & & & \\
 & & & & & & & & & & & 1 & & & \rho_{10} & \\
\hline
 & & & -\rho_8 & & & -\rho_6 & & & & \rho_6 & & \rho_3 & & & \\
 & & & & & & & \rho_9 & & & & & & 1 & & \\
 & & & & & & & & & & & \rho_9 & & & 1 & \\
 & & & & & & & & & & & & & & & \rho_1
\end{array}\right)
$$

(Up to a similarity transformation) Shastry's R-matrix (1986), not of difference form!

Algebraic approach

Expressions for the Boltzmann weights

$$\rho_1(\lambda,\mu) = \cos(\lambda)\cos(\mu)e^{h-l} + \sin(\lambda)\sin(\mu)e^{l-h}$$

$$\rho_4(\lambda,\mu) = \cos(\lambda)\cos(\mu)e^{l-h} + \sin(\lambda)\sin(\mu)e^{h-l}$$

$$\rho_3(\lambda,\mu) = \frac{\cos(\lambda)\cos(\mu)e^{h-l} - \sin(\lambda)\sin(\mu)e^{l-h}}{\cos^2(\lambda) - \sin^2(\mu)}$$

$$\rho_5(\lambda,\mu) = \frac{\cos(\lambda)\cos(\mu)e^{l-h} - \sin(\lambda)\sin(\mu)e^{h-l}}{\cos^2(\lambda) - \sin^2(\mu)}$$

$$\rho_6(\lambda,\mu) = \frac{\operatorname{sh}(2(h-l))}{2u(\cos^2(\lambda) - \sin^2(\mu))}$$

$$\rho_7(\lambda,\mu) = \rho_4(\lambda,\mu) - \rho_5(\lambda,\mu)$$

$$\rho_8(\lambda,\mu) = \rho_1(\lambda,\mu) - \rho_3(\lambda,\mu)$$

$$\rho_9(\lambda,\mu) = \sin(\lambda)\cos(\mu)e^{l-h} - \cos(\lambda)\sin(\mu)e^{h-l}$$

$$\rho_{10}(\lambda,\mu) = \sin(\lambda)\cos(\mu)e^{h-l} - \cos(\lambda)\sin(\mu)e^{l-h}$$

The parameters λ, μ, h and l are subject to the constraints

$$\frac{\operatorname{sh}(2h)}{\sin(2\lambda)} = \frac{\operatorname{sh}(2l)}{\sin(2\mu)} = u$$

Expressions for the Boltzmann weights

$$\rho_1(\lambda,\mu) = \cos(\lambda)\cos(\mu)e^{h-l} + \sin(\lambda)\sin(\mu)e^{l-h}$$

$$\rho_4(\lambda,\mu) = \cos(\lambda)\cos(\mu)e^{l-h} + \sin(\lambda)\sin(\mu)e^{h-l}$$

$$\rho_3(\lambda,\mu) = \frac{\cos(\lambda)\cos(\mu)e^{h-l} - \sin(\lambda)\sin(\mu)e^{l-h}}{\cos^2(\lambda) - \sin^2(\mu)}$$

$$\rho_5(\lambda,\mu) = \frac{\cos(\lambda)\cos(\mu)e^{l-h} - \sin(\lambda)\sin(\mu)e^{h-l}}{\cos^2(\lambda) - \sin^2(\mu)}$$

$$\rho_6(\lambda,\mu) = \frac{\mathrm{sh}(2(h-l))}{2u(\cos^2(\lambda) - \sin^2(\mu))}$$

$$\rho_7(\lambda,\mu) = \rho_4(\lambda,\mu) - \rho_5(\lambda,\mu)$$

$$\rho_8(\lambda,\mu) = \rho_1(\lambda,\mu) - \rho_3(\lambda,\mu)$$

$$\rho_9(\lambda,\mu) = \sin(\lambda)\cos(\mu)e^{l-h} - \cos(\lambda)\sin(\mu)e^{h-l}$$

$$\rho_{10}(\lambda,\mu) = \sin(\lambda)\cos(\mu)e^{h-l} - \cos(\lambda)\sin(\mu)e^{l-h}$$

The parameters λ, μ, h and l are subject to the constraints

$$\frac{\mathrm{sh}(2h)}{\sin(2\lambda)} = \frac{\mathrm{sh}(2l)}{\sin(2\mu)} = u$$

Remarks

- Shastry's R-matrix is not of difference form and does not fit into the usual classification of rational, trigonometric and elliptic R-matrices
- Accordingly no uniform parameterization is known (setting $\mu = 0$ we find that the corresponding L-matrix lives on a Riemann surface of genus 3)
- Shastry's R-matrix can be constructed by gluing together two free fermion R-matrices

$$\check{R}(\lambda) = \begin{pmatrix} \cos(\lambda) & & & \\ & 1 & x\sin(\lambda) & \\ & x^{-1}\sin(\lambda) & 1 & \\ & & & \cos(\lambda) \end{pmatrix}$$

Algebraic approach

Shastry's R-matrix is not a very convenient tool for calculations. The algebraic Bethe ansatz for the corresponding vertex model (Martins and Ramos 97, 98) is partially conjectural. However, the algebraic approach was useful for

- The construction of the quantum transfer matrix approach to the thermodynamics of the Hubbard model (Jüttner, Klümper, Suzuki 98)
- The construction of a boost operator (Links et al. 2001)
- The algebraic construction of the eigenstates on the infinite interval and the clarification of the role of the Yangian (S. Murakami and FG 97, 98)
- The generalization to higher rank building blocks (Maassarani 98, Peng and Yue 02)

Yangian symmetry at n = 0

Since the Hubbard model describes electrons we better use a graded version $+--+$ of the Yang-Baxter algebra,

$$\check{R}(\lambda,\mu)\big(\mathcal{T}(\lambda)\otimes_s\mathcal{T}(\mu)\big)=\big(\mathcal{T}(\mu)\otimes_s\mathcal{T}(\lambda)\big)\check{R}(\lambda,\mu)$$

where

$$\mathcal{T}(\lambda)=\begin{pmatrix} D_1^1(\lambda) & C_1^1(\lambda) & C_2^1(\lambda) & D_2^1(\lambda) \\ B_1^1(\lambda) & A_1^1(\lambda) & A_2^1(\lambda) & B_2^1(\lambda) \\ B_1^2(\lambda) & A_1^2(\lambda) & A_2^2(\lambda) & B_2^2(\lambda) \\ D_1^2(\lambda) & C_1^2(\lambda) & C_2^2(\lambda) & D_2^2(\lambda) \end{pmatrix}$$

consists of four 2×2 blocks.

Yangian symmetry at n = 0

Since the Hubbard model describes electrons we better use a graded version $+--+$ of the Yang-Baxter algebra,

$$\check{R}(\lambda,\mu)\big(\mathcal{T}(\lambda)\otimes_s\mathcal{T}(\mu)\big) = \big(\mathcal{T}(\mu)\otimes_s\mathcal{T}(\lambda)\big)\check{R}(\lambda,\mu)$$

where

$$\mathcal{T}(\lambda) = \begin{pmatrix} D_1^1(\lambda) & C_1^1(\lambda) & C_2^1(\lambda) & D_2^1(\lambda) \\ B_1^1(\lambda) & A_1^1(\lambda) & A_2^1(\lambda) & B_2^1(\lambda) \\ B_1^2(\lambda) & A_1^2(\lambda) & A_2^2(\lambda) & B_2^2(\lambda) \\ D_1^2(\lambda) & C_1^2(\lambda) & C_2^2(\lambda) & D_2^2(\lambda) \end{pmatrix}$$

consists of four 2×2 blocks.

Thermodynamic limit with respect to $|0\rangle$ possible on the level of operators (à la Faddeev and Sklyanin 78) This requires regularization of the monodromy matrix. Let $V(\lambda) = \langle 0|\mathcal{L}_m(\lambda)|0\rangle$ and $V^{(2)}(\lambda,\mu) = \langle 0|\mathcal{L}_m(\lambda)\otimes_s \mathcal{L}_m(\mu)|0\rangle$. Then V regularizes $\mathcal{T}$ and $V^{(2)}$ regularizes $\mathcal{T}\otimes\mathcal{T}$. Since $V^{(2)}\neq V\otimes V$, the R-matrix is changed (simplified!).

Since the Hubbard model describes electrons we better use a graded version $+--+$ of the Yang-Baxter algebra,

$$\check{R}(\lambda,\mu)\big(\mathcal{T}(\lambda)\otimes_s\mathcal{T}(\mu)\big)=\big(\mathcal{T}(\mu)\otimes_s\mathcal{T}(\lambda)\big)\check{R}(\lambda,\mu)$$

where

$$\mathcal{T}(\lambda)=\begin{pmatrix} D_1^1(\lambda) & C_1^1(\lambda) & C_2^1(\lambda) & D_2^1(\lambda)\\ B_1^1(\lambda) & A_1^1(\lambda) & A_2^1(\lambda) & B_2^1(\lambda)\\ B_1^2(\lambda) & A_1^2(\lambda) & A_2^2(\lambda) & B_2^2(\lambda)\\ D_1^2(\lambda) & C_1^2(\lambda) & C_2^2(\lambda) & D_2^2(\lambda)\end{pmatrix}$$

consists of four 2×2 blocks.

Thermodynamic limit with respect to $|0\rangle$ possible on the level of operators (à la Faddeev and Sklyanin 78) This requires regularization of the monodromy matrix. Let $V(\lambda)=\langle 0|\mathcal{L}_m(\lambda)|0\rangle$ and $V^{(2)}(\lambda,\mu)=\langle 0|\mathcal{L}_m(\lambda)\otimes_s\mathcal{L}_m(\mu)|0\rangle$. Then V regularizes $\mathcal{T}$ and $V^{(2)}$ regularizes $\mathcal{T}\otimes\mathcal{T}$. Since $V^{(2)}\neq V\otimes V$, the R-matrix is changed (simplified!).

- The regularized monodromy matrix has the 'integral representation'

$$\tilde{\mathcal{T}}(\lambda)=I_4+\sum_m(\tilde{\mathcal{L}}_m(\lambda)-I_4)+\sum_{m>n}(\tilde{\mathcal{L}}_m(\lambda)-I_4)(\tilde{\mathcal{L}}_n(\lambda)-I_4)+\dots$$

where $\tilde{\mathcal{L}}_j(\lambda)=V(\lambda)^{-j-1}\mathcal{L}_j(\lambda)V(\lambda)^j$

Yangian symmetry at n = 0

Since the Hubbard model describes electrons we better use a graded version $+--+$ of the Yang-Baxter algebra,

$$\check{R}(\lambda,\mu)\big(\mathcal{T}(\lambda)\otimes_s \mathcal{T}(\mu)\big) = \big(\mathcal{T}(\mu)\otimes_s \mathcal{T}(\lambda)\big)\check{R}(\lambda,\mu)$$

where

$$\mathcal{T}(\lambda) = \begin{pmatrix} D_1^1(\lambda) & C_1^1(\lambda) & C_2^1(\lambda) & D_2^1(\lambda) \\ B_1^1(\lambda) & A_1^1(\lambda) & A_2^1(\lambda) & B_2^1(\lambda) \\ B_1^2(\lambda) & A_1^2(\lambda) & A_2^2(\lambda) & B_2^2(\lambda) \\ D_1^2(\lambda) & C_1^2(\lambda) & C_2^2(\lambda) & D_2^2(\lambda) \end{pmatrix}$$

consists of four 2×2 blocks.

Thermodynamic limit with respect to $|0\rangle$ possible on the level of operators (à la Faddeev and Sklyanin 78) This requires regularization of the monodromy matrix. Let $V(\lambda) = \langle 0|\mathcal{L}_m(\lambda)|0\rangle$ and $V^{(2)}(\lambda,\mu) = \langle 0|\mathcal{L}_m(\lambda)\otimes_s \mathcal{L}_m(\mu)|0\rangle$. Then V regularizes $\mathcal{T}$ and $V^{(2)}$ regularizes $\mathcal{T}\otimes\mathcal{T}$. Since $V^{(2)} \neq V\otimes V$, the R-matrix is changed (simplified!).

- The regularized monodromy matrix has the 'integral representation'

$$\tilde{\mathcal{T}}(\lambda) = I_4 + \sum_m (\tilde{\mathcal{L}}_m(\lambda) - I_4) + \sum_{m>n} (\tilde{\mathcal{L}}_m(\lambda) - I_4)(\tilde{\mathcal{L}}_n(\lambda) - I_4) + \dots$$

where $\tilde{\mathcal{L}}_j(\lambda) = V(\lambda)^{-j-1}\mathcal{L}_j(\lambda)V(\lambda)^j$

- Submatrix $A(\lambda)$ decouples

$$\check{r}(\lambda,\mu)\big(A(\lambda)\otimes A(\mu)\big) = \big(A(\mu)\otimes A(\lambda)\big)\check{r}(\lambda,\mu)$$

Yangian symmetry at n = 0

Since the Hubbard model describes electrons we better use a graded version $+--+$ of the Yang-Baxter algebra,

$$\check{R}(\lambda,\mu)\big(\mathcal{T}(\lambda)\otimes_s\mathcal{T}(\mu)\big) = \big(\mathcal{T}(\mu)\otimes_s\mathcal{T}(\lambda)\big)\check{R}(\lambda,\mu)$$

where

$$\mathcal{T}(\lambda) = \begin{pmatrix} D_1^1(\lambda) & C_1^1(\lambda) & C_2^1(\lambda) & D_2^1(\lambda) \\ B_1^1(\lambda) & A_1^1(\lambda) & A_2^1(\lambda) & B_2^1(\lambda) \\ B_1^2(\lambda) & A_1^2(\lambda) & A_2^2(\lambda) & B_2^2(\lambda) \\ D_1^2(\lambda) & C_1^2(\lambda) & C_2^2(\lambda) & D_2^2(\lambda) \end{pmatrix}$$

consists of four 2×2 blocks.

Thermodynamic limit with respect to $|0\rangle$ possible on the level of operators (à la Faddeev and Sklyanin 78) This requires regularization of the monodromy matrix. Let $V(\lambda) = \langle 0|\mathcal{L}_m(\lambda)|0\rangle$ and $V^{(2)}(\lambda,\mu) = \langle 0|\mathcal{L}_m(\lambda)\otimes_s\mathcal{L}_m(\mu)|0\rangle$. Then V regularizes $\mathcal{T}$ and $V^{(2)}$ regularizes $\mathcal{T}\otimes\mathcal{T}$. Since $V^{(2)}\neq V\otimes V$, the R-matrix is changed (simplified!).

- The regularized monodromy matrix has the 'integral representation'

$$\tilde{\mathcal{T}}(\lambda) = I_4 + \sum_m (\tilde{\mathcal{L}}_m(\lambda) - I_4) + \sum_{m>n} (\tilde{\mathcal{L}}_m(\lambda) - I_4)(\tilde{\mathcal{L}}_n(\lambda) - I_4) + \ldots$$

where $\tilde{\mathcal{L}}_j(\lambda) = V(\lambda)^{-j-1}\mathcal{L}_j(\lambda)V(\lambda)^j$

- Submatrix $A(\lambda)$ decouples

$$\check{r}(\lambda,\mu)\big(A(\lambda)\otimes A(\mu)\big) = \big(A(\mu)\otimes A(\lambda)\big)\check{r}(\lambda,\mu)$$

- Moreover the change of variables

$$v(\lambda) = -\mathrm{i}\,\mathrm{ctg}\,(2\lambda)\,\mathrm{ch}(2h)$$

transforms $\check{r}(\lambda,\mu)$ into the rational R-matrix of the XXX spin chain,

$$\check{r}(\lambda,\mu) = \frac{2\mathrm{i}u + (v(\lambda)-v(\mu))P}{2\mathrm{i}u + v(\lambda) - v(\mu)}$$

Yangian symmetry at n = 0

- It follows that the coefficients J_n^0, J_n^α in the asymptotic expansion

$$A(\lambda) = I_2 + 2iu \sum_{n=1}^{\infty} \frac{J_{n-1}^0 I_2 + J_{n-1}^\alpha \sigma^\alpha}{v(\lambda)^n}$$

generate a representation of the Yangian Y(gl(2))

Yangian symmetry at n = 0

- It follows that the coefficients J_n^0, J_n^α in the asymptotic expansion

$$A(\lambda) = I_2 + 2iu \sum_{n=1}^{\infty} \frac{J_{n-1}^0 I_2 + J_{n-1}^\alpha \sigma^\alpha}{v(\lambda)^n}$$

  generate a representation of the Yangian Y(gl(2))

- The centre of this algebra is obtained by expanding the quantum determinant,

$$\begin{aligned} \det_q(A(\lambda)) &= A_1^1(\lambda) A_2^2(\check{\lambda}) - A_2^1(\lambda) A_1^2(\check{\lambda}) \\ &= 1 + 2iu \sum_{n=1}^{\infty} \frac{a_{n-1}}{v(\lambda)^n} \end{aligned}$$

  where $\check{\lambda}$ is determined by the condition that $v(\check{\lambda}) = v(\lambda) - 2iu$.

Yangian symmetry at n = 0

- It follows that the coefficients J_n^0, J_n^α in the asymptotic expansion

$$A(\lambda) = I_2 + 2iu \sum_{n=1}^{\infty} \frac{J_{n-1}^0 I_2 + J_{n-1}^\alpha \sigma^\alpha}{v(\lambda)^n}$$

  generate a representation of the Yangian Y(gl(2))

- The centre of this algebra is obtained by expanding the quantum determinant,

$$\begin{aligned} \det_q(A(\lambda)) &= A_1^1(\lambda)A_2^2(\check{\lambda}) - A_2^1(\lambda)A_1^2(\check{\lambda}) \\ &= 1 + 2iu \sum_{n=1}^{\infty} \frac{a_{n-1}}{v(\lambda)^n} \end{aligned}$$

  where $\check{\lambda}$ is determined by the condition that $v(\check{\lambda}) = v(\lambda) - 2iu$.

- The asymptotic expansions can be calculated term by term

Yangian symmetry at n = 0

- It follows that the coefficients J_n^0, J_n^α in the asymptotic expansion

$$A(\lambda) = I_2 + 2iu \sum_{n=1}^{\infty} \frac{J_{n-1}^0 I_2 + J_{n-1}^\alpha \sigma^\alpha}{v(\lambda)^n}$$

  generate a representation of the Yangian Y(gl(2))
- The centre of this algebra is obtained by expanding the quantum determinant,

$$\begin{aligned} \det_q(A(\lambda)) &= A_1^1(\lambda) A_2^2(\check{\lambda}) - A_2^1(\lambda) A_1^2(\check{\lambda}) \\ &= 1 + 2iu \sum_{n=1}^{\infty} \frac{a_{n-1}}{v(\lambda)^n} \end{aligned}$$

  where $\check{\lambda}$ is determined by the condition that $v(\check{\lambda}) = v(\lambda) - 2iu$.
- The asymptotic expansions can be calculated term by term
- Explicit expressions for the zeroth and first level Yangian generators,

$$J_0^\alpha = \sum_j S_j^\alpha \, ,$$

$$J_1^\alpha = -\tfrac{i}{4} \sum_j \left(S_{j\,j+1}^\alpha - S_{j\,j-1}^\alpha \right) + 2u \sum_{j<k} \varepsilon^{\alpha\beta\gamma} S_j^\beta S_k^\gamma$$

Yangian symmetry at n = 0

- It follows that the coefficients J_n^0, J_n^α in the asymptotic expansion

$$A(\lambda) = I_2 + 2iu \sum_{n=1}^{\infty} \frac{J_{n-1}^0 I_2 + J_{n-1}^\alpha \sigma^\alpha}{v(\lambda)^n}$$

  generate a representation of the Yangian Y(gl(2))
- The centre of this algebra is obtained by expanding the quantum determinant,

$$\begin{aligned} \det_q(A(\lambda)) &= A_1^1(\lambda) A_2^2(\check\lambda) - A_2^1(\lambda) A_1^2(\check\lambda) \\ &= 1 + 2iu \sum_{n=1}^{\infty} \frac{a_{n-1}}{v(\lambda)^n} \end{aligned}$$

  where $\check\lambda$ is determined by the condition that $v(\check\lambda) = v(\lambda) - 2iu$.
- The asymptotic expansions can be calculated term by term
- Explicit expressions for the zeroth and first level Yangian generators,

$$J_0^\alpha = \sum_j S_j^\alpha \, ,$$

$$J_1^\alpha = -\tfrac{i}{4} \sum_j \left(S_{j\,j+1}^\alpha - S_{j\,j-1}^\alpha \right) + 2u \sum_{j<k} \varepsilon^{\alpha\beta\gamma} S_j^\beta S_k^\gamma$$

- The $\frac{1}{v(\lambda)}$-expansion of $\det_q(A(\lambda))$

$$a_0 = 0 \, , \quad a_1 = iH/2$$

  where H is the Hubbard Hamiltonian

Yangian symmetry at n = 0

- It follows that the coefficients J_n^0, J_n^α in the asymptotic expansion

$$A(\lambda) = I_2 + 2iu \sum_{n=1}^{\infty} \frac{J_{n-1}^0 I_2 + J_{n-1}^\alpha \sigma^\alpha}{v(\lambda)^n}$$

  generate a representation of the Yangian Y(gl(2))
- The centre of this algebra is obtained by expanding the quantum determinant,

$$\begin{aligned} \det_q(A(\lambda)) &= A_1^1(\lambda)A_2^2(\check{\lambda}) - A_2^1(\lambda)A_1^2(\check{\lambda}) \\ &= 1 + 2iu \sum_{n=1}^{\infty} \frac{a_{n-1}}{v(\lambda)^n} \end{aligned}$$

  where $\check{\lambda}$ is determined by the condition that $v(\check{\lambda}) = v(\lambda) - 2iu$.
- The asymptotic expansions can be calculated term by term
- Explicit expressions for the zeroth and first level Yangian generators,

$$J_0^\alpha = \sum_j S_j^\alpha,$$

$$J_1^\alpha = -\tfrac{i}{4} \sum_j \left(S_{jj+1}^\alpha - S_{jj-1}^\alpha\right) + 2u \sum_{j<k} \varepsilon^{\alpha\beta\gamma} S_j^\beta S_k^\gamma$$

- The $\frac{1}{v(\lambda)}$-expansion of $\det_q(A(\lambda))$

$$a_0 = 0, \quad a_1 = iH/2$$

  where H is the Hubbard Hamiltonian
- Two pairs of normalized creation operators

$$\begin{aligned} F_a(\lambda)^\dagger &= -ie^h \cos(\lambda) C_a^1(\lambda) D_1^1(\lambda)^{-1} \\ Z^a(\lambda)^\dagger &= (-1)^{3-a} ie^{-h} \cos(\lambda) \\ &\qquad B_2^{3-a}(\lambda) D_2^2(\lambda)^{-1} \end{aligned}$$

  for $a = 1, 2$, spin-up, spin-down

Yangian symmetry at n = 0

- Annihilation operators $F_a(\lambda)$, $Z^a(\lambda)$ similar
- Commutation relations between the normalized operators follow from the YBA. For $\lambda \neq \mu \pmod{2\pi}$

$$F_a(\lambda)^\dagger F_b(\mu)^\dagger = -F_c(\mu)^\dagger F_d(\lambda)^\dagger \check{r}^{cd}_{ab}(\lambda,\mu)$$

$$F_a(\lambda) F_b(\mu)^\dagger = -F_c(\mu)^\dagger F_d(\lambda) \check{r}^{ca}_{db}(\mu,\lambda)$$

$$Z^a(\lambda)^\dagger Z^b(\mu)^\dagger = -\check{r}^{ab}_{cd}(\mu,\lambda) Z^c(\mu)^\dagger Z^d(\lambda)^\dagger$$

$$Z^a(\lambda) Z^b(\mu)^\dagger = -\check{r}^{db}_{ca}(\lambda,\mu) Z^c(\mu)^\dagger Z^d(\lambda)$$

$$F_a(\lambda)^\dagger Z^b(\mu)^\dagger = -Z^b(\mu)^\dagger F_a(\lambda)^\dagger$$

$$F_a(\lambda) Z^b(\mu)^\dagger = -Z^b(\mu)^\dagger F_a(\lambda)$$

$F_a(\lambda)$, $F_a(\lambda)^\dagger$ and $Z^a(\lambda)$, $Z^a(\lambda)^\dagger$ are forming (right and left) representations of the (graded) Faddeev-Zamolodchikov algebra with two-particle S-matrix $\check{r}(\lambda,\mu)$. All operators are odd. The algebra guarantees the factorization of the N-particle S-matrix into products of two-particle S-matrices.

Yangian symmetry at n = 0

- Annihilation operators $F_a(\lambda)$, $Z^a(\lambda)$ similar
- Commutation relations between the normalized operators follow from the YBA. For $\lambda \neq \mu \pmod{2\pi}$

$$F_a(\lambda)^\dagger F_b(\mu)^\dagger = -F_c(\mu)^\dagger F_d(\lambda)^\dagger \check{r}^{cd}_{ab}(\lambda,\mu)$$

$$F_a(\lambda) F_b(\mu)^\dagger = -F_c(\mu)^\dagger F_d(\lambda) \check{r}^{ca}_{db}(\mu,\lambda)$$

$$Z^a(\lambda)^\dagger Z^b(\mu)^\dagger = -\check{r}^{ab}_{cd}(\mu,\lambda) Z^c(\mu)^\dagger Z^d(\lambda)^\dagger$$

$$Z^a(\lambda) Z^b(\mu)^\dagger = -\check{r}^{db}_{ca}(\lambda,\mu) Z^c(\mu)^\dagger Z^d(\lambda)$$

$$F_a(\lambda)^\dagger Z^b(\mu)^\dagger = -Z^b(\mu)^\dagger F_a(\lambda)^\dagger$$

$$F_a(\lambda) Z^b(\mu)^\dagger = -Z^b(\mu)^\dagger F_a(\lambda)$$

$F_a(\lambda)$, $F_a(\lambda)^\dagger$ and $Z^a(\lambda)$, $Z^a(\lambda)^\dagger$ are forming (right and left) representations of the (graded) Faddeev-Zamolodchikov algebra with two-particle S-matrix $\check{r}(\lambda,\mu)$. All operators are odd. The algebra guarantees the factorization of the N-particle S-matrix into products of two-particle S-matrices.

- Transformation under the action of the Yangian

$$[J_0^\alpha, F_a(\lambda)^\dagger] = \tfrac{1}{2} F_b(\lambda)^\dagger \sigma^\alpha_{ba}$$

$$[J_1^\alpha, F_a(\lambda)^\dagger] = -\tfrac{1}{2} \sin k(\lambda) F_b(\lambda)^\dagger \sigma^\alpha_{ba} + u\,\varepsilon^{\alpha\beta\gamma} F_b(\lambda)^\dagger \sigma^\beta_{ba} J_0^\gamma$$

$$[J_0^\alpha, Z^a(\lambda)^\dagger] = \tfrac{1}{2} Z^b(\lambda)^\dagger \sigma^\alpha_{ba}$$

$$[J_1^\alpha, Z^a(\lambda)^\dagger] = -\tfrac{1}{2} \sin p(\lambda) Z^b(\lambda)^\dagger \sigma^\alpha_{ba} - u\,\varepsilon^{\alpha\beta\gamma} Z^b(\lambda)^\dagger \sigma^\beta_{ba} J_0^\gamma$$

It follows that n-particle states transform like tensor products of evaluation representations of the Yangian

Yangian symmetry at n = 0

- Annihilation operators $F_a(\lambda)$, $Z^a(\lambda)$ similar
- Commutation relations between the normalized operators follow from the YBA. For $\lambda \neq \mu \pmod{2\pi}$

$$F_a(\lambda)^\dagger F_b(\mu)^\dagger = -F_c(\mu)^\dagger F_d(\lambda)^\dagger \check{r}^{cd}_{ab}(\lambda,\mu)$$

$$F_a(\lambda) F_b(\mu)^\dagger = -F_c(\mu)^\dagger F_d(\lambda) \check{r}^{ca}_{db}(\mu,\lambda)$$

$$Z^a(\lambda)^\dagger Z^b(\mu)^\dagger = -\check{r}^{ab}_{cd}(\mu,\lambda) Z^c(\mu)^\dagger Z^d(\lambda)^\dagger$$

$$Z^a(\lambda) Z^b(\mu)^\dagger = -\check{r}^{db}_{ca}(\lambda,\mu) Z^c(\mu)^\dagger Z^d(\lambda)$$

$$F_a(\lambda)^\dagger Z^b(\mu)^\dagger = -Z^b(\mu)^\dagger F_a(\lambda)^\dagger$$

$$F_a(\lambda) Z^b(\mu)^\dagger = -Z^b(\mu)^\dagger F_a(\lambda)$$

$F_a(\lambda)$, $F_a(\lambda)^\dagger$ and $Z^a(\lambda)$, $Z^a(\lambda)^\dagger$ are forming (right and left) representations of the (graded) Faddeev-Zamolodchikov algebra with two-particle S-matrix $\check{r}(\lambda,\mu)$. All operators are odd. The algebra guarantees the factorization of the N-particle S-matrix into products of two-particle S-matrices.

- Transformation under the action of the Yangian

$$[J_0^\alpha, F_a(\lambda)^\dagger] = \tfrac{1}{2} F_b(\lambda)^\dagger \sigma^\alpha_{ba}$$

$$[J_1^\alpha, F_a(\lambda)^\dagger] = -\tfrac{1}{2}\sin k(\lambda) F_b(\lambda)^\dagger \sigma^\alpha_{ba} + u\,\varepsilon^{\alpha\beta\gamma} F_b(\lambda)^\dagger \sigma^\beta_{ba} J_0^\gamma$$

$$[J_0^\alpha, Z^a(\lambda)^\dagger] = \tfrac{1}{2} Z^b(\lambda)^\dagger \sigma^\alpha_{ba}$$

$$[J_1^\alpha, Z^a(\lambda)^\dagger] = -\tfrac{1}{2}\sin p(\lambda) Z^b(\lambda)^\dagger \sigma^\alpha_{ba} - u\,\varepsilon^{\alpha\beta\gamma} Z^b(\lambda)^\dagger \sigma^\beta_{ba} J_0^\gamma$$

  It follows that n-particle states transform like tensor products of evaluation representations of the Yangian

- Construction of the creation and annihilation operators of bound states also possible, yield the bare S-matrix for the scattering of 'strings' of any length, strings are Yangian singlet

Yangian symmetry at n = 0

Using the commutation relations for the elements of the monodromy matrix we obtain for our bound state operators

$$F^{(2m)}(\lambda_i)^\dagger F^{(2n)}(\mu_j)^\dagger = \frac{\zeta-\eta+(n+m)\mathrm{i}u}{\zeta-\eta-(n+m)\mathrm{i}u}\,\frac{\zeta-\eta+|n-m|\mathrm{i}u}{\zeta-\eta-|n-m|\mathrm{i}u}$$

$$\prod_{s=1}^{\min\{m,n\}-1}\left[\frac{\zeta-\eta+(n+m-2s)\mathrm{i}u}{\zeta-\eta-(n+m-2s)\mathrm{i}u}\right]^2 F^{(2n)}(\mu_j)^\dagger F^{(2m)}(\lambda_i)^\dagger$$

$$F^{(2m)}(\lambda_i)^\dagger F_a(\mu)^\dagger = \frac{\zeta-\sin k(\mu)+mi u}{\zeta-\sin k(\mu)-mi u} F_a(\mu)^\dagger F^{(2m)}(\lambda_i)^\dagger\,, \quad a=1,2$$

where ζ is the centre of the $2m$-string and η the centre of the $2n$-string. We interpret these relations as Faddeev-Zamolodchikov algebra. Here particles without internal degrees of freedom are involved. The bound-state bound-state S-matrix in is of the same form as for the scattering of bound states of magnons in the XXX-chain (P. Kulish and F. Smirnov 1982, 85).
Yangian Singlet:

$$[J_0^\alpha, F^{(2m)}(\lambda_i)^\dagger] = [J_1^\alpha, F^{(2m)}(\lambda_i)^\dagger] = 0$$

A summary and open questions

- Because of its important applications in condensed matter physics the Hubbard model is one of the best-studied integrable models. It is a paradigm in condensed matter physics, since it describes a generic deviation from the one-particle picture, which explains the ubiquitious occurence of anti-ferromagnetism in nature and the existence of Mott insulators.
- Arguably, it is also a paradigmatic integrable system. It contains the Heisenberg chain and the Gaudin model as limiting cases. It shows most of the difficulties that possibly exist in (Yang-Baxter) integrable systems (nested Bethe ansatz, R-matrix not of difference form, no simple Lie algebra symmetry).
- Those properties (relating to the spectrum) which can be obtained from the Lieb-Wu equations have been obtained.

A summary and open questions

- Because of its important applications in condensed matter physics the Hubbard model is one of the best-studied integrable models. It is a paradigm in condensed matter physics, since it describes a generic deviation from the one-particle picture, which explains the ubiquitious occurence of anti-ferromagnetism in nature and the existence of Mott insulators.
- Arguably, it is also a paradigmatic integrable system. It contains the Heisenberg chain and the Gaudin model as limiting cases. It shows most of the difficulties that possibly exist in (Yang-Baxter) integrable systems (nested Bethe ansatz, R-matrix not of difference form, no simple Lie algebra symmetry).
- Those properties (relating to the spectrum) which can be obtained from the Lieb-Wu equations have been obtained.

Some open questions

- Simpler and more complete algebraic Bethe ansatz
- Better understanding of the meaning and structure of Shastry's R-matrix, generalization for fixed dimension (Alcaraz and Bariev 1999)?
- Role of the Yangian symmetry at finite density?