Quantum corrections to the string Bethe ansatz †

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Introduction

The AdS/CFT correspondence:

The large N limit of $\mathcal{N} = 4$ Yang-Mills is dual to type IIB string theory on $AdS_5 \times S^5$

$\Downarrow$

Spectra of both theories should agree

$\rightarrow$ Difficult to test, because the correspondence is a strong/weak coupling duality: we can not use perturbation theory on both sides

String energies expanded at large λ

$$E(\lambda) = \lambda^{1/4} E_0 + \lambda^{-1/4} E_1 + \lambda^{-3/4} E_2 + \ldots$$

Scaling dimensions of gauge operators at small λ

$$\Delta(\lambda) = D_0 + \lambda D_1 + \lambda^2 D_2 + \ldots$$

$$\boxed{E(\lambda) \leftrightarrow \Delta(\lambda)}$$

$\rightarrow$ **Integrability illuminates both sides of the correspondence**

$\rightarrow$ S_{string} **should interpolate to** S_{gauge}

Integrability in the AdS/CFT correspondence

A complete formulation of the AdS/CFT correspondence

$$\Downarrow$$

Precise identification of **string states**
with local **gauge invariant operators**

$$E\sqrt{\alpha'} = \Delta$$

$\longrightarrow$ There is strong evidence in the **supergravity regime**,

$$\underline{R^2 \gg \alpha'} \qquad (R^4 = 4\pi g_{YM}^2 N \alpha'^2)$$

Difficulties:
- String quantization in $AdS_5 \times S^5$
- Obtaining the whole spectrum of $\mathcal{N} = 4$ is truly involved

An insight: There is a maximally supersymmetric **plane-wave background** for the IIB string [Blau et al]

$$\Downarrow$$

Allows quantization in the light-cone gauge [Metsaev, Tseytlin]

Plane-wave geometry $\Rightarrow$ **Penrose limit**

The Penrose limit shows up on the field theory side [Berenstein, Maldacena, Nastase]

$\Downarrow$

Operators carrying **large charges**, $\boxed{\text{tr}\,(X_1^J \ldots), J \gg 1}$

$\longrightarrow$ Dual description in terms of **small closed strings** whose center moves with angular momentum J along a circle in S^5 [Gubser, Klebanov, Polyakov]

Generalization: Operators of the form tr $(X_1^{J_1} X_2^{J_2} X_3^{J_3})$ are dual to strings with three angular momenta J_i [Frolov, Tseytlin]

$\Rightarrow$ The energy of these **semiclassical strings** admits an analytic expansion in λ/J^2

$$E = J\left[1 + c_1\left(\frac{J_i}{J}\right)\frac{\lambda}{J^2} + \ldots\right]$$

$\Downarrow$

Suggests a comparison with the anomalous dimensions of large Yang-Mills operators:

- Bare dimension $\Delta_0 \rightarrow J$
- **One-loop** anomalous dimension $\rightarrow \frac{\lambda}{J} c_1 \left(\frac{J_i}{J}\right)$

Verifying AdS/CFT in large spin sectors

$\Downarrow$

Computation of the anomalous dimensions of large operators

(Difficult problem due to **operator mixing**)

Insightful solution:

→ The **one-loop planar dilatation operator** of $\mathcal{N}=4$ Yang-Mills leads to an integrable spin chain ($SO(6)$ in the scalar sector [Minahan,Zarembo] or $PSU(2,2|4)$ in the complete theory [Beisert,Staudacher])

The Bethe ansatz

→ The rapidities u_j parameterizing the momenta of the magnons satisfy a set of **Bethe equations**

$$e^{ip_j J} \equiv \left(\frac{u_j + i/2}{u_j - i/2}\right)^J = \prod_{k\neq j}^{M} \frac{u_j - u_k + i}{u_j - u_k - i} \equiv \prod_{k\neq j}^{M} S(u_j, u_k)$$

Thermodynamic limit: **integral equations**

→ Assuming integrability an **asymptotic all loop Bethe ansatz** has been proposed [Beisert,Dippel,Staudacher]

The quantum string Bethe ansatz

String non-linear sigma model on the coset

$$\frac{PSU(2,2|4)}{SO(4,1)\times SO(5)}$$

Integrable

[Mandal,Suryanarayana,Wadia;Bena,Polchinski,Roiban]

Admits a Lax representation: there is a family of connections $A(z)$ **flat** for all values of the spectral parameter z

$$dA(z) - A(z)\wedge A(z) = 0$$

(Flatness of $A(z)$ is equivalent to flatness of $J = -g^{-1}dg$ and conservation of the Noether current K)

$$\Downarrow$$

Classical solutions of the sigma model are parametrized by an integral equation

[Kazakov,Marshakov,Minahan,Zarembo]

$$-\frac{x}{x^2-\frac{\lambda}{16\pi^2J^2}}\frac{\Delta}{J} + 2\pi k \;=\; 2\int_{\mathcal{C}} dx'\frac{\rho(x')}{x-x'} \qquad x\in\mathcal{C}$$

Reminds of the **thermodynamic Bethe equations** for the spin chain ...

In fact, it **leads to the spin chain equations** when $\lambda/J^2\to 0$

The previous string integral equations are
classical/thermodynamic equations

$\Downarrow$

Assuming integrability survives at the quantum level,
a discretization would provide
a **quantum string Bethe ansatz**

$\longrightarrow$ There is an even greater similarity between the **classical string Bethe ansatz** and the **long range Bethe ansatz** for the gauge theory of [Beisert,Dippel,Staudacher]

$\Downarrow$

After some convenient map

gauge: $$2 \fint_{\mathcal{C}} dx' \frac{\rho_g(x')}{x-x'} = \frac{1}{x} \frac{1}{1-\frac{\lambda}{2J^2x^2}} + \frac{\lambda}{J^2} \frac{1}{x} \int dx' \frac{\rho_g(x')}{1-\frac{\lambda}{J^2xx'}} + 2\pi k$$

string: $$2 \fint_{\mathcal{C}} dx' \frac{\rho_s(x')}{x-x'} = \frac{1}{x} \frac{1}{1-\frac{\lambda}{2J^2x^2}} + \frac{\lambda}{J^2} \frac{1}{x} \int dx' \frac{\rho_s(x')}{1-\frac{\lambda}{J^2x^2}} + 2\pi k$$

The S-matrices of the (discrete) quantum string and the long range gauge Bethe ansätze differ simply by a phase [Arutyunov,Frolov,Staudacher]

$$\boxed{S_{st}(p_j, p_k) = e^{i\,\theta(p_j,p_k)} S_g(p_j, p_k)}$$

$$\theta(p_j, p_k) = 2\sum_{r=2}^{\infty} c_r(\lambda) \left(\frac{\lambda}{16\pi^2}\right)^r \Big(q_r(p_j)q_{r+1}(p_k) - q_{r+1}(p_j)q_r(p_k)\Big)$$

($q_r(p)$ are the conserved charges of the integrable system)

→ To recover the integrable structure of the classical string the coefficients must satisfy $c_r(\lambda) \to 1$ as $\lambda \to \infty$

→ **This phase should interpolate from the string to the gauge theory (strong weak/coupling interpolation)**

$\Downarrow$

Explicit form of $c_r(\lambda)$

To constrain **the string Bethe ansatz** and find the structure of the dressing phase we can compare to **one-loop corrections** to semiclassical strings

One-loop corrections to semiclassical strings

One-loop corrections are obtained from the spectrum of **quadratic fluctuations** around a classical solution
[Frolov,Tseytlin;Frolov,Park,Tseytlin]

$$E_1 = \sum_{n=-\infty}^{\infty} e(n)$$

$\rightarrow e(n)$ is a sum over bosonic and fermionic frequencies with mode number n

In the simpler case, $SU(2)$ with $k = 2m$,

$$e(n) = \sqrt{1 + \frac{(n + \sqrt{n^2 - 4m^2})^2}{4(\mathcal{J}^2 + m^2)}} + \sqrt{1 + \frac{n^2 - 2m^2}{\mathcal{J}^2 + m^2}} + 2\sqrt{1 + \frac{n^2}{\mathcal{J}^2 + m^2}} - 4\sqrt{1 + \frac{n^2 - m^2}{\mathcal{J}^2 + m^2}}$$

$\Downarrow$

Bosonic fluctuations along S^3 + remaining S^5 + AdS_5 directions + fermionic fluctuations

($\rightarrow$ Analogous, but much more involved expression, in the $SL(2)$ sector)

From a careful analysis of the sum
[Schäfer-Nameki,Zamaklar,Zarembo]

→ **Agreement up to order** $\lambda^3/J^6 \equiv 1/\mathcal{J}^6$

→ **Disagreement** if $c_r(\lambda) = 1$, because of **non-analytic terms** in λ

Origin of the **non-analytic terms**

$\Downarrow$

Difficulties in the evaluation of
$E_1 = \sum_{n=-\infty}^{\infty} e(n)$

→ Expanding $e(n)$ for fixed n at large $\mathcal{J}$: divergences at high n [Schäfer-Nameki,Zamaklar,Zarembo]
⇒ Cannot reach the high energy spectrum

→ Expanding $e(n)$ at fixed $x = n/\mathcal{J}$: regular at large x, but divergences at $x = 0$
⇒ Cannot reach the lowest modes of the spectrum

Solution: Combine both expansions

[Beisert,Tseytlin; Schäfer-Nameki]

$$e(n) = e_1(n) + e_2(n/\mathcal{J})$$

$e_1(n)$ and $e_2(n/\mathcal{J})$ are the regular terms for fixed n and $n/\mathcal{J}$

($e_1(n)$ is zeta regularized, and $e_2(n)$ substracting negative powers of x)

$\rightarrow \sum e_1$ contains only $(1/\mathcal{J})^{2n}$ powers

$\rightarrow \sum e_2$ leads to $(1/\mathcal{J})^{2n+1}$ powers

For instance, for $SL(2)$ circular strings

$$\int_{-\infty}^{\infty} \mathcal{J}\, dx\, e_2^{SL(2)}(x) = -\frac{(k-m)^3\, m^3}{3\mathcal{J}^5}\left(1 - \frac{3k^2 - 8km}{2\mathcal{J}^2} + \ldots\right)$$

Corrections to the string ansatz

In order to **cure the disagreement**, and fit the first non-analytic term a **quantum correction** to the string Bethe ansatz was suggested [Beisert,Tseytlin]

$$\theta(p_j, p_k) = 2\sum_{r=2}^{\infty} c_r(\lambda)\left(\frac{\lambda}{16\pi^2}\right)^r \Big(q_r(p_j)q_{r+1}(p_k) - q_{r+1}(p_j)q_r(p_k)\Big)$$

$$\text{with} \quad c_2(\lambda) = 1 - \frac{16}{3}\frac{1}{\sqrt{\lambda}}$$

Then the energy shift for the **one-loop string correction**

$$\delta E_{\text{one-loop}} = -\frac{(k-m)^3\, m^3}{3\mathcal{J}^5} + \mathcal{O}(1/\mathcal{J}^7)$$

agrees with the quantum string Bethe ansatz computation!!

The negative correction term opens the possibility that $c_r(\lambda)$ could **interpolate** between the strong coupling value

$$c_2(\infty) = 1$$

and zero at weak coupling

$$\Downarrow$$

Suggests a solution to the three-loop discrepancy!

But this is simply the first coefficient in the dressing phase ...

Let us now recall the explicit expansion of the **first quantum correction to the rotating string ...**

→ In the $SU(2)$ **sector with** $k = 2m$ the energy shift for the one-loop correction

$$\delta E_{SU(2)} = -\frac{m^6}{3\,\mathcal{J}^5} + \frac{m^8}{3\,\mathcal{J}^7} - \frac{49\,m^{10}}{120\,\mathcal{J}^9} + \frac{2\,m^{12}}{5\,\mathcal{J}^{11}} - \frac{5749\,m^{14}}{13440\,\mathcal{J}^{13}} + \;\ldots$$

→ In the $SL(2)$ **case for general** k **and** m

$$\delta E_{SL(2)} = -\frac{(k-m)^3\,m^3}{3\,\mathcal{J}^5}\left[1 - \frac{P_2}{2\,\mathcal{J}^2} + \frac{P_4}{40\,\mathcal{J}^4} - \frac{P_6}{80\,\mathcal{J}^6} + \frac{P_8}{4480\,\mathcal{J}^8} + \ldots\right]$$

with P_n homegeneous polynomials

$$\begin{aligned}
P_2 &= 3k^2 - 8km\,,\\
P_4 &= 75k^4 - 455k^3m + 679k^2m^2 - 153km^3 + 29m^4\,,\\
P_6 &= 175k^6 - 1755k^5m + 5635k^4m^2 - 6843k^3m^3\\
&\quad + \; 2823k^2m^4 - 562km^5 + 2m^6\\
P_8 &= 11025k^8 - 159565k^7m + 820785k^6m^2\\
&\quad - \; 1923509k^5m^3 + 2159033k^4m^4 - 1141813k^3m^5\\
&\quad + \; 303665k^2m^6 - 31753km^7 + 2557m^8
\end{aligned}$$

... and compare to the quantum string Bethe ansatz

(At order $1/\mathcal{J}^{r+s}$: polynomial with $r+s-4$ coefficients, enough to fix the $(r+s-3)/2$ terms in the phase)

Careful comparison with the one-loop string correction requires a slightly more general ansatz [Beisert,Klose]

$$\theta(p_j,p_k) = 2\sum_{r=2}^{\infty}\sum_{s=r+1}^{\infty} c_{r,s}(\lambda)\left(\frac{\lambda}{16\pi^2}\right)^{\frac{r+s-1}{2}}\Big(q_r(p_j)q_s(p_k) - q_s(p_j)q_r(p_k)\Big)$$

It reminds to solve the $(SL(2))$ **corrected Bethe equation**

$$\begin{aligned} 2\,⨍_{\mathcal{C}} dy\frac{\rho(y)}{x-y} &= 2\pi\,k_i - \frac{x}{x^2-(1/4\pi\mathcal{J})^2}\Bigg[1-\left(\frac{1}{4\pi\mathcal{J}}\right)^2\int_{\mathcal{C}} dy\,\frac{2\rho(y)}{yx} \\ &\quad - 2a_{r,s}\frac{1}{\sqrt{\lambda}}\left(\frac{1}{4\pi\mathcal{J}}\right)^{r+s-1}\int_{\mathcal{C}} dy\,\rho(y)\left(\frac{1}{x^{r-1}y^s}-\frac{1}{x^{s-1}y^r}\right)\Bigg] \end{aligned}$$

The coefficients [RH,López]

$$\begin{aligned} c_{r,s} &= \delta_{r+1,s} + \frac{1}{\sqrt{\lambda}}a_{r,s} \\ a_{r,s} &= -8\,\frac{(r-1)(s-1)}{(r+s-2)(s-r)} \end{aligned}$$

fit the quantum string result **up to the order we checked:** $1/\mathcal{J}^{101}$ **in the** $SU(2)$ **sector and** $1/\mathcal{J}^{13}$ **in the** $SL(2)$ **sector!!**

→ The coefficients are **universal**: valid in all sectors

(Remain valid in the $SU(3)$ sector [Freyhult,Kristjansen])

First quantum correction: first step towards the reconstruction of the **complete** S**-matrix**

Constraints on the dressing factor

- The structure of the complete S-matrix is [Beisert]

$$S = S_0(p_1, p_2).\left[\hat{S}_{SU(2|2)} \otimes \hat{S}_{SU(2|2)}\right]$$

  - The term in the bracket is determined by the **symmetries**: Yang-Baxter
  - The coefficient is the dressing factor: constrained by unitarity and crossing ($\rightarrow$ **dynamics**) [Janik]

$$\theta(x_1, x_2) + \theta(1/x_1, x_2) = -2i \log h(x_1, x_2) \ ,$$

with

$$h(x_1, x_2) = \frac{x_2^-}{x_2^+} \frac{x_1^- - x_2^+}{x_1^+ - x_2^+} \frac{1 - 1/x_1^- x_2^-}{1 - 1/x_1^+ x_2^-}$$

$\rightarrow$ $h(x_1, x_2)$ contains information on the dynamics of S_{gauge}

$\rightarrow$ $\theta(x_1, x_2)$ contains information on the quantum structure of the string

An expansion of both sides has been shown to agree, using the explicit form of the one-loop correction in $\theta(x_j, x_k)$ [Arutyunov,Frolov]

$\Downarrow$

The coefficients $c_{r,s}(\lambda)$ are a solution of the crossing equations

Conclusions

- Testing AdS/CFT in large spin sectors $\Rightarrow$ Integrability in the planar limit of $\mathcal{N}=4$ Yang-Mills
  - Precision tests of the correspondence
- Quantum corrections to classical strings constrain the string Bethe ansatz
  - Simple form of the first correction
  - An explicit solution to the crossing equation has recently been found [Beisert]

◦ A proof of the the AdS/CFT correspondence requires identification of spectra, together with interpolation as the coupling evolves

$$\Downarrow$$

The dressing factor should interpolate from the string to the gauge theory, and strong to weak coupling

$$S_{st}(p_j, p_k) = e^{i\theta(p_j, p_k)} S_g(p_j, p_k)$$

◦ Algebraic origin of the structure of the dressing phase