On-Shell Methods in Field Theory

including work with
C. Berger, Z. Bern, M. Czakon, L. Dixon, D. Dunbar, D. Forde, R. Roiban, V. Smirnov

Marcel Grossman Meeting/Integrability Session, July 27, 2006

- You're here to hear about
  - Integrability
  - Gravity
  - Duality
- My motivation is from a *completely* different source
- But the tools I'll present are useful for both

Next-to-Leading Order for the LHC

If we're going to win the pennant, we've got to start thinking we're not as good as we think we are — Casey Stengel

- Reliable QCD predictions
  - Detailed understanding of backgrounds
  - Measurement of luminosity
  - Prepare for *measurement* of new physics, not just discovery

Everything at a hadron collider involves QCD

- On-shell methods: the unitarity-bootstrap approach
  - High-multiplicity targets
  - Recycling: analytic results desirable
  - Amplitudes rather than helicity-summed cross sections

Physical Quantities

- Paradigm: differential cross section for scattering of physical states — S matrix element
- Integrated quantities: jet cross sections, anomalous dimensions, operator matrix elements, …
- A physical quantity is finite
  - Definition in terms of physical coupling
  - Infrared safe
  - Computation could in principle be finite at every stage of computation

Where Is the Simplicity?

Simplicity is the ultimate sophistication — Leonardo da Vinci

- Simpler results should come out of simpler calculations
- Why don't traditional Feynman-diagram techniques do that?
  - Propagators and vertices involve off-shell states ($p^2 \neq 0$). These aren't gauge invariant.
  - Involve non-physical helicities
  - Add up large number of gauge non-invariant contributions to obtain gauge-invariant result; but gauge invariance is not manifest
  - Compute a huge amount of redundant and unnecessary information
  - Factorial growth in number of terms whereas optimal complexity is probably only polynomial per helicity

On-Shell Methods

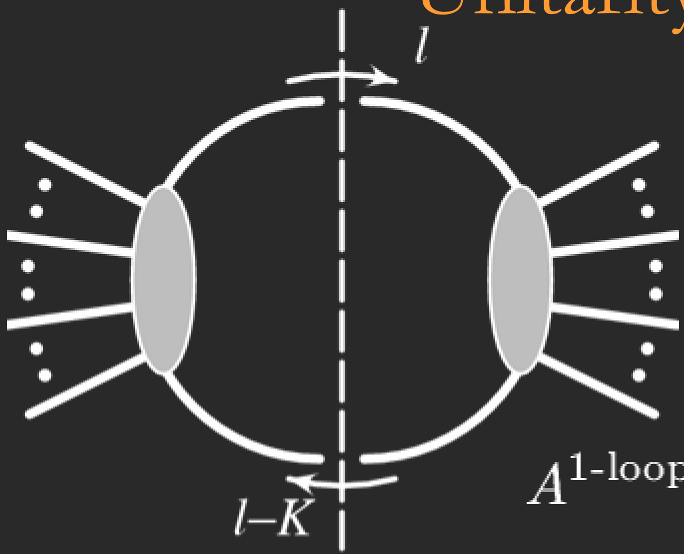
- To make the simplicity manifest, we should make sure all steps are done in terms of on-shell, gauge-invariant states
- Use of properties of amplitudes as tools to compute them
  - Spinor helicity & twistors $\Leftarrow$ kinematics
  - Factorization $\Leftarrow$ trees
  - Unitarity $\Leftarrow$ loops
  - Underlying Feynman-integral representation
- Especially powerful in theories with redundant covariant variables, like gauge theories and gravity

Unitarity-Based Method

- Use a general property of amplitudes as a practical tool for computing them
- Sew loop amplitudes out of on-shell tree amplitudes: summation of Cutkosky relation
- Use knowledge of possible Feynman integrals (field theory origin) & all modern techniques: identities, modern reduction techniques, differential equations, reduction to master integrals, etc.
- Can sew more than two tree amplitudes: generalized unitarity
- QCD = $\mathcal{N}=4 \; + \; \delta\mathcal{N}=1 \; + \; \delta\mathcal{N}=0$
- Gravity ~ $\text{QCD}^2 = (\mathcal{N}=4 \; + \; \delta\mathcal{N}=1 \; + \; \delta\mathcal{N}=0)^2_{\text{sym}}$

Kawai, Lewellen, & Tye (1987)

Unitarity-Based Calculations

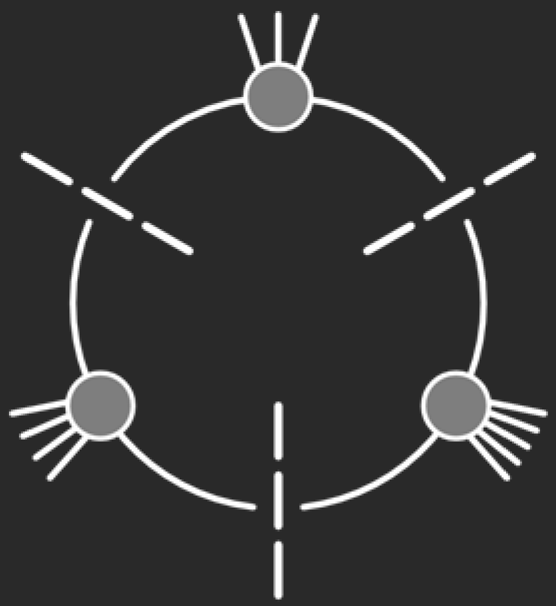


– Bern, Dixon, Dunbar, & DAK (1994)

$$A^{\text{1-loop}} = \sum_{\text{cuts}\ K^2} \int \frac{d^{4-2\epsilon}\ell}{(2\pi)^{4-2\epsilon}} \frac{i}{\ell^2} A^{\text{tree}}_{\text{left}} \frac{i}{(\ell-K)^2} A^{\text{tree}}_{\text{right}}$$

- At one loop in D=4 for SUSY ⇒ full answer
- For non-SUSY theory, must work in D=4-2є ⇒ full answer
  van Neerven (1986): dispersion relations converge
- Merge channels rather than blindly summing: find function w/given cuts in all channels
- No tensor reductions: problem reduced to computing rational coefficients of known basis integrals

Generalized Cuts



- Isolate contributions of smaller set of integrals, at higher loops as well

Unitarity and Maximal Supersymmetry

- In $\mathcal{N}=4$ gauge theory, it's been used to
  - compute infinite series of amplitudes (all-multiplicity);
  - compute four- and five-point amplitudes and discover an iteration relation,

$$M_n^{(2)}(\epsilon) = \frac{1}{2}\left(M_n^{(1)}(\epsilon)\right)^2 + f^{(2)}(\epsilon)\, M_n^{(1)}(2\epsilon) - \frac{1}{2}\zeta_2^2$$

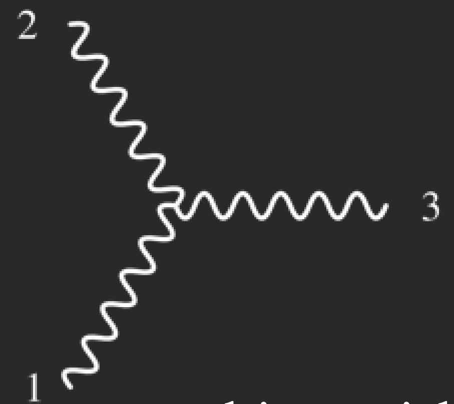
$$f^{(2)}(\epsilon) = -(\zeta_2 + \zeta_3\epsilon + \zeta_4\epsilon^2 + \cdots)$$

- …and is being used to
  - compute the four-loop anomalous dimension
- In $\mathcal{N}=8$ supergravity, it's been used to
  - demonstrate that divergences are delayed to at least five loops, contrary to prior conventional wisdom

Bern, Dixon, Dunbar, Perelstein, Rozowsky (1998),
presented at Strings '99 in Potsdam

(confirmed: Howe & Stelle (2002))

A Practical Lesson From Twistors



- On-shell with real momenta this vanishes

$$p_1 \cdot p_2 = p_2 \cdot p_3 = p_3 \cdot p_1 = 0$$

- For real momenta, $\tilde{\lambda} = \pm\bar{\lambda}$, so all spinor products vanish too
- For *complex* momenta $\tilde{\lambda} \neq \pm\bar{\lambda}$

$$\Rightarrow \quad \langle i\, j \rangle = 0 \quad \text{or} \quad [i\, j] = 0$$

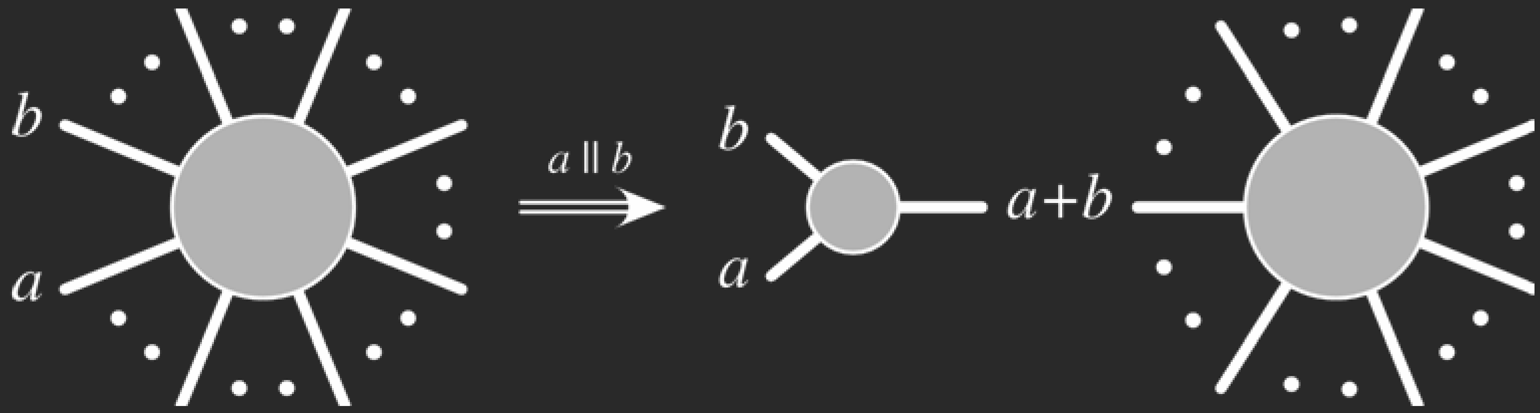
but *not* necessarily both!

Witten (2003)

Factorization

The best way to have a good idea is to have a lot of ideas — Linus Pauling

- Also a general property of field-theory amplitudes



- Constrain terms to have correct collinear limit: factorization as a calculational tool
  - Used in $Z \to q\bar{q}gg$ to obtain simple form for rational terms

Bern, Dixon, DAK (1997)

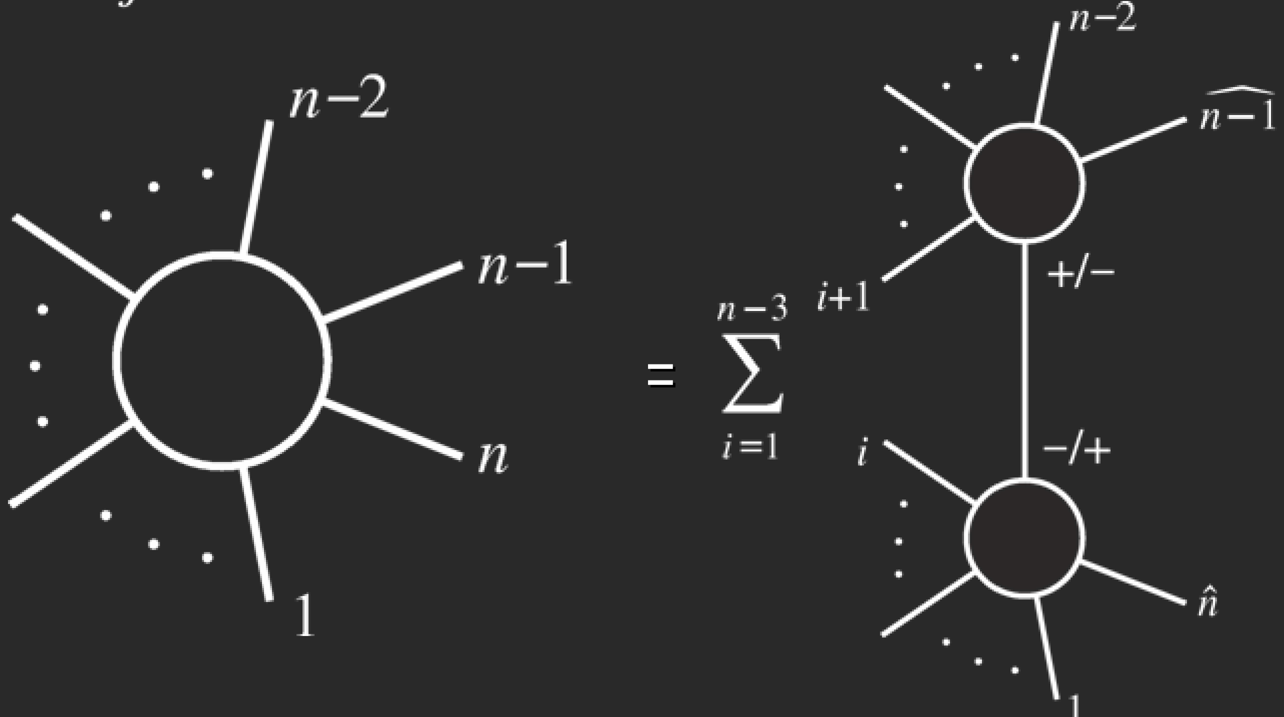
- Can this be made systematic?

On-Shell Recursion Relations

Britto, Cachazo, Feng (2004); & Witten (1/2005)

- Amplitudes written as sum over 'factorizations' into *on-shell* amplitudes — but evaluated for *complex* momenta

$$A_n \sim \sum_j A_{j+1}(\ldots, -\hat{K}_{1\ldots j}) \frac{1}{K^2_{1\ldots j}} A_{n-j+1}(\hat{K}_{1\ldots j}, \ldots)$$



Proof Ingredients

Less is more. My architecture is almost nothing — Mies van der Rohe

Britto, Cachazo, Feng, Witten (2005)

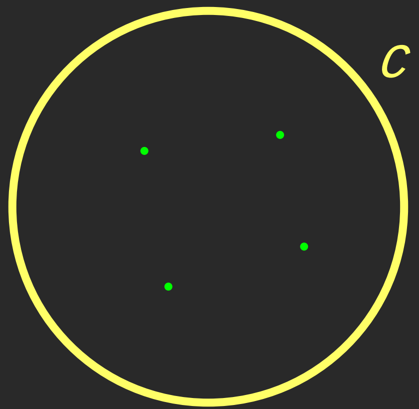
- Complex shift of momenta (j, l)

$$p_j^\mu \to p_j^\mu(z) = p_j^\mu - \frac{z}{2}\langle j^- | \gamma^\mu | l^- \rangle,$$

$$p_l^\mu \to p_l^\mu(z) = p_l^\mu + \frac{z}{2}\langle j^- | \gamma^\mu | l^- \rangle.$$

- Behavior as z → ∞: need $A(z) \to 0$
- Basic complex analysis
- Knowledge of factorization: at tree level, tracks known multiparticle-pole and collinear factorization

Proof



- Consider the contour integral

$$\frac{1}{2\pi i}\oint_C \frac{dz}{z} A(z)$$

- Determine A(0) in terms of other poles

$$A(0) = -\sum_{\text{poles } \alpha} \operatorname*{Res}_{z=z_\alpha} \frac{A(z)}{z}$$

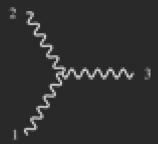
- Poles determined by knowledge of factorization in invariants
- At tree level

$$A(0) = \sum_{i,j,h} \frac{A_L^h(z=z_{ij}) A_R^{-h}(z=z_{ij})}{P_{ij}^2}$$

- Very general: relies only on complex analysis + factorization
- Applied to gravity

Bedford, Brandhuber, Spence, & Travaglini (2/2005)
Cachazo & Svrček (2/2005)

⇒ Everything derivable from a three–vertex!



- Massive amplitudes

Badger, Glover, Khoze, Svrček (4/2005, 7/2005)
Forde & DAK (7/2005)

- Integral coefficients

Bern, Bjerrum-Bohr, Dunbar, & Ita (7/2005)

- Connection to Cachazo–Svrček–Witten construction

Risager (8/2005)

- CSW construction for gravity ⇒ Twistor string for $\mathcal{N}=8$?

Bjerrum-Bohr, Dunbar, Ita, Perkins, & Risager (9/2005)
Abou-Zeid, Hull, & Mason (6/2005)

On-Shell Recursion at Loop Level

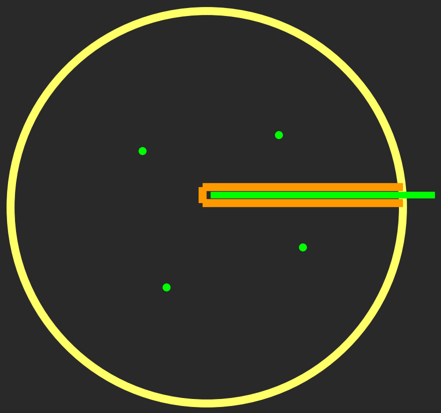
Bern, Dixon, DAK (1–7/2005)

- Complex shift of momenta (j, l)

$$p_j^\mu \to p_j^\mu(z) = p_j^\mu - \frac{z}{2}\langle j^- | \gamma^\mu | l^- \rangle,$$
$$p_l^\mu \to p_l^\mu(z) = p_l^\mu + \frac{z}{2}\langle j^- | \gamma^\mu | l^- \rangle.$$

- Behavior as $z \to \infty$: require $A(z) \to 0$
- Basic complex analysis: treat branch cuts
- Knowledge of *complex* factorization:
  - at tree level, tracks known factorization for real momenta
  - at loop level, there are subtleties: double poles $\frac{[a\, b]}{\langle a\, b\rangle^2}$ and 'unreal' poles $\frac{[a\, b]}{\langle a\, b\rangle}$ arise

Derivation



- Consider the contour integral

$$\frac{1}{2\pi i}\oint_C \frac{dz}{z}\,A(z)$$

- Determine A(0) in terms of other poles and branch cuts

$$A(0) = -\sum_{\text{poles }\alpha} \underset{z=z_\alpha}{\text{Res}} \frac{A(z)}{z} - \int_{\text{Branch}} \frac{dz}{z}\,\text{Disc}_B A(z)$$

Rational terms

Cut terms

⇑

From unitarity

- Recursion gives

$$\text{Recursive} = -\sum_{\text{poles }\alpha} \operatorname*{Res}_{z=z_\alpha} \frac{\text{Rational}[\hat{C}(z)]}{z} - \sum_{\text{poles }\alpha} \operatorname*{Res}_{z=z_\alpha} \frac{\hat{R}(z)}{z}$$

Double-counted: ‘overlap’

- Subtract it off

$$A(0) = c_\Gamma\left[\hat{C}(0) + \text{Recursive} + \sum_{\text{poles }\alpha} \operatorname*{Res}_{z=z_\alpha} \frac{\text{Rational}[\hat{C}(z)]}{z}\right]$$

Compute explicitly from known $\hat{C}$

A New 2→4 QCD Amplitude

$$A_6(1^-,2^-,3^-,4^+,5^+,6^+)$$

$$= c_\Gamma\left[(V_6^g + 4V_6^f + V_6^s)A_6^{\rm tree} + i(4F_6^f + F_6^s)\right.$$

$$\left. -\frac{n_f}{N_c}\left(A_6^{\rm tree}(V_6^s + V_6^f) + i(F_6^s + F_6^f)\right)\right]$$

Only rational terms missing

Bern, Dixon, Dunbar, & DAK (1994);
Bidder, Bjerrum-Bohr, Dixon, & Dunbar (2004);
Bern, Bjerrum-Bohr, Dunbar, & Ita (2005)

$$A_6(1^-, 2^-, 3^-, 4^+, 5^+, 6^+)$$

$$\widehat{R}_6 = \widehat{R}_6^a + \widehat{R}_6^a\Big|_{\text{flip 1}}$$

$$\begin{aligned}\widehat{R}_6^a \;=\; & \frac{i}{6}\frac{1}{[2\,3]\,\langle 5\,6\rangle\;\langle 5^-|\,(3+4)\,|2^-\rangle}\Bigg\{-\frac{[4\,6]^3\,[2\,5]\,\langle 5\,6\rangle}{[1\,2]\,[3\,4]\,[6\,1]} - \frac{\langle 1\,3\rangle^3\,\langle 2\,5\rangle\,[2\,3]}{\langle 3\,4\rangle\,\langle 4\,5\rangle\,\langle 6\,1\rangle} \\ & + \frac{\langle 1^-|\,(2+3)\,|4^-\rangle^2}{[3\,4]\,\langle 6\,1\rangle}\Big(\frac{\langle 1^-|\,2\,|4^-\rangle - \langle 1^-|\,5\,|4^-\rangle}{s_{234}} + \frac{\langle 1\,3\rangle}{\langle 3\,4\rangle} - \frac{[4\,6]}{[6\,1]}\Big) \\ & - \frac{\langle 1\,3\rangle^2(3\,\langle 1^-|\,2\,|4^-\rangle + \langle 1^-|\,3\,|4^-\rangle)}{\langle 3\,4\rangle\,\langle 6\,1\rangle} + \frac{[4\,6]^2(3\,\langle 1^-|\,5\,|4^-\rangle + \langle 1^-|\,6\,|4^-\rangle)}{[3\,4]\,[6\,1]}\Bigg\}\end{aligned}$$

$$X(1,2,3,4,5,6)\Big|_{\text{flip 1}} \equiv X(3,2,1,6,5,4)$$

All-Multiplicity Amplitudes

We can then write the result for the unrenormalized amplitude $A_{n,s}^{\text{1-loop}} = V^s A_n^{\text{tree}} + iF^s$ in the following form,

$$V_n^s A_n^{\text{tree}} + iF_n^s = c_\Gamma \left[\hat{C}_n + \hat{R}_n\right] + \frac{1}{3} A_n^{\mathcal{N}=1 \text{ chiral}} + \frac{2}{9} A_n^{\text{tree}} \tag{1}$$

where $\hat{C}_n$ are the cut-containing contributions computed in ref. [ ], completed so as to remove $s_1 \to s_2$ spurious singularities,

$$\hat{C}_n = -\frac{1}{3 s_{12}^3} A^{\text{tree}}(1^-, 2^-, 3^+, \ldots, n^+) \times \sum_{j=4}^{n-1} \frac{L_2\big((-s_{2\cdots(m-1)})/(-s_{2\cdots m})\big)}{s_{2\cdots m}^3} \operatorname{tr}_+\left[\not k_1 \not k_2 \not k_m \not K_{m\cdots 1}\right] \operatorname{tr}_+\left[\not k_1 \not k_2 \not K_{m\cdots 1} \not k_m\right] \operatorname{tr}_+\left[\not k_1 \not k_2 (\not k_m \not K_{m\cdots 1} - \not K_{m\cdots 1} \not k_m)\right] \tag{2}$$

The computations then yield,

$$\begin{aligned} \hat{R}_n(1^-, 2^-, 3^+, \ldots, n^+) = \frac{1}{3} A^{\text{tree}}(1^-, 2^-, 3^+, \ldots, n^+) \times \sum_{i_1=1}^{n-4} \Bigg(\sum_{i_2=i_1+3}^{n-1} \Big[& C_1(n;i_1,i_2)\Big(T_1(n;i_1,i_2,i_2) + T_1(n;i_1,i_2,i_2+1)\Big) \\ & + C_2(n;i_1,i_2)\Big(T_{2a}(n;i_1,i_2) + T_{2b}(n;i_1,i_2)\Big) \\ & + C_3(n;i_1,i_2)\Big(T_{3a}(n;i_1,i_2) + T_{3b}(n;i_1,i_2) + T_{3c}(n;i_1,i_2)\Big)\Big] + T_4(n;i_1)\Bigg) \end{aligned} \tag{3}$$

In this equation,

$$\begin{aligned} C_1(n;i_1,i_2) &= \frac{\langle (i_1+1)\,(i_1+2)\rangle}{\langle 1^- | \not K_{(i_2+1)\cdots n} \not K_{(i_1+2)\cdots i_2} | (i_1+1)^+\rangle \langle 1^- | \not K_{(i_2+1)\cdots n} \not K_{(i_1+3)\cdots i_2} | (i_1+2)^+\rangle}, \\ C_2(n;i_1,i_2) &= \frac{\langle i_2\,(i_2+1)\rangle\, C_1(n;i_1,i_2)}{s_{(i_1+2)\cdots i_2} \langle 1^- | \not K_{2\cdots(i_1+1)} \not K_{(i_1+2)\cdots i_2} | (i_2+1)^+\rangle \langle 1^- | \not K_{2\cdots(i_1+1)} \not K_{(i_1+2)\cdots(i_2-1)} | i_2^+\rangle}, \\ C_3(n;i_1,i_2) &= s_{(i_1+2)\cdots i_2}^4\, C_2(n;i_1,i_2). \end{aligned} \tag{4}$$

The terms T_i are given by,

$$T_1(n;i_1,i_2,j) = \frac{s_{(i_1+2)\cdots i_2} \langle 1\,j\rangle \langle 1^- | \not K_{(i_2+1)\cdots n} \not K_{(i_1+2)\cdots i_2} | j^+\rangle \langle 1^- | \not K_{2\cdots i_2} \not K_{(i_1+2)\cdots i_2} (\not k_j \not K_{2\cdots(j-1)} - \not K_{(i_1+2)\cdots(j-1)} \not k_j) | 1^+\rangle}{2 \langle 1^- | \not K_{2\cdots(i_1+1)} \not K_{(i_1+2)\cdots i_2} | j^+\rangle^2}; \tag{5}$$

(Note that $T_1(n;i_1,n-1,n) = 0$.)

$$T_{2a}(n;i_1,i_2) = \sum_{l=(i_1+2)}^{i_2} \langle 1^- | \not K_{(i_2+1)\cdots n} \not K_{(i_1+2)\cdots i_2} | l^+\rangle\, f_1(n;l,i_1,i_2); \tag{6}$$

$$\begin{aligned} T_{2b}(n;i_1,i_2) = -\sum_{l=(i_1+3)}^{i_2-1} \sum_{p=l+1}^{i_2} & \frac{f_2(n;l,p;i_1,i_2)}{\langle 1^- | \not K_{(i_2+1)\cdots n} \not K_{(i_1+2)\cdots i_2} \not K_{(i_1+2)\cdots(l-1)} \not K_{l\cdots(p-1)} | p^+\rangle} \\ & \times \frac{\langle (l-1)\,l\rangle \langle 1^- | \not K_{(i_2+1)\cdots n} \not K_{(i_1+2)\cdots i_2} \not K_{l\cdots p} \not K_{(i_1+2)\cdots p} \not K_{(i_1+2)\cdots i_2} \not K_{2\cdots i_2} | 1^+\rangle^3}{s_{l\cdots p} \langle 1^- | \not K_{(i_2+1)\cdots n} \not K_{(i_1+2)\cdots i_2} \not K_{(i_1+2)\cdots p} \not K_{l\cdots p} | (l-1)^+\rangle} \\ & \times \frac{\langle 1^- | \not K_{2\cdots i_2} \not K_{(i_1+2)\cdots i_2} \not K_{(i_1+2)\cdots(l-1)} [\mathcal{F}(l,p)]^2 \not K_{(i_1+2)\cdots p} \not K_{(i_1+2)\cdots i_2} \not K_{2\cdots i_2} | 1^+\rangle}{\langle 1^- | \not K_{(i_2+1)\cdots n} \not K_{(i_1+2)\cdots i_2} \not K_{(i_1+2)\cdots p} \not K_{(l+1)\cdots p} | l^+\rangle}; \end{aligned} \tag{7}$$

$$T_{3a}(n;i_1,i_2) = \sum_{l=i_2+1}^{n-1} \frac{\langle 1\,l\rangle \langle 1\,(l+1)\rangle \langle 1^- | \not K_{l(l+1)} \not K_{(l+1)\cdots n} | 1^+\rangle}{\langle l\,(l+1)\rangle}; \tag{8}$$

$$T_{3b}(n;i_1,i_2) = \frac{\langle 1^- | \not K_{2\cdots i_2} | (i_2+1)^-\rangle \langle 1^- | \not K_{2\cdots(i_1+1)} \not K_{2\cdots i_2} | 1^+\rangle \langle 1\,(i_2+1)\rangle^2}{\langle 1^- | \not K_{2\cdots(i_1+1)} \not K_{(i_1+2)\cdots i_2} | (i_2+1)^+\rangle}; \tag{9}$$

$$\begin{aligned} T_{3c}(n;i_1,i_2) = \sum_{l=i_2+1}^{n-2} \sum_{p=l+1}^{n-1} & \frac{\langle 1^- | \not K_{l\cdots p} \not K_{(p+1)\cdots n} | 1^+\rangle^3}{\langle 1^- | \not K_{(l+1)\cdots n} \not K_{(l-1)\cdots p} | l^+\rangle} \\ & \times \frac{\langle p\,(p+1)\rangle \langle 1^- | \not K_{2\cdots(l-1)} [\mathcal{F}(l,p)]^2 \not K_{(p+1)\cdots n} | 1^+\rangle\, f_3(n;l,p,i_1,i_2)}{s_{l\cdots p} \langle 1^- | \not K_{2\cdots(l-1)} \not K_{l\cdots(p-1)} | p^+\rangle \langle 1^- | \not K_{2\cdots(l-1)} \not K_{l\cdots p} | (p+1)^+\rangle}; \end{aligned} \tag{10}$$

$$T_4(n;i_1) = -\frac{[(i_1+2)\,(i_1+3)]\,\langle (i_1+3)\,1\rangle}{2 \langle 1^- | \not K_{2\cdots(i_1+1)} | (i_1+2)^-\rangle}. \tag{11}$$

The f_i appearing in the above equations are given by,

$$f_1(n;l,i_1,i_2) = \begin{cases} -s_{(i_1+2)\cdots i_2}^2 \langle 1^- | \not K_{(i_1+2)\cdots i_2} \not K_{2\cdots(i_1+1)} | 1^+\rangle \times \dfrac{\langle 1^- | \not K_{2\cdots i_2} \not K_{(i_1+2)\cdots(i_2-1)} | i_2^+\rangle \langle i_2^+ | \not K_{2\cdots(i_2-1)} | 1^+\rangle}{\langle 1^- | \not K_{2\cdots(i_1+1)} \not K_{(i_1+2)\cdots(i_2-1)} | i_2^+\rangle}, & l = i_2 \\ \langle 1^- | \not K_{2\cdots i_2} \not K_{(i_1+2)\cdots i_2} | (l+1)^+\rangle \times \dfrac{\langle 1^- | \not K_{2\cdots i_2} \not K_{(i_1+2)\cdots i_2} \not K_{l(l+1)} \not K_{(i_1+2)\cdots l} \not K_{(i_1+2)\cdots i_2} \not K_{2\cdots i_2} | 1^+\rangle}{\langle l\,(l+1)\rangle}, & (i_1+2) \le l < i_2 \end{cases} \tag{12}$$

$$f_2(n;l,p,i_1,i_2) = \begin{cases} \dfrac{\langle i_2^- | \not K_{(i_1+2)\cdots i_2} \not K_{2\cdots(i_1+1)} | 1^+\rangle}{s_{(i_1+2)\cdots i_2} \langle 1^- | \not K_{(i_2+1)\cdots n} \not K_{l\cdots i_2} \not K_{(i_1+2)\cdots(l-1)} \not K_{2\cdots(i_1+1)} | 1^+\rangle}, & p = i_2 \\ \dfrac{\langle p\,(p+1)\rangle}{\langle 1^- | \not K_{(i_2+1)\cdots n} \not K_{(i_1+2)\cdots i_2} \not K_{(i_1+2)\cdots(l-1)} \not K_{l\cdots p} | (p+1)^+\rangle}, & l+1 \le p < i_2 \end{cases} \tag{13}$$

$$f_3(n;l,p,i_1,i_2) = \begin{cases} \dfrac{\langle 1^- | \not K_{2\cdots(i_1+1)} \not K_{(i_1+2)\cdots i_2} | (i_2+1)^+\rangle}{\langle 1^- | \not K_{(p+1)\cdots n} \not K_{(i_2+1)\cdots p} \not K_{(i_1+2)\cdots i_2} \not K_{2\cdots(i_1+1)} | 1^+\rangle}, & l = i_2+1 \\ \dfrac{\langle (l-1)\,l\rangle}{\langle 1^- | \not K_{(p+1)\cdots n} \not K_{l\cdots p} | (l-1)^+\rangle}, & l > i_2+1 \end{cases} \tag{14}$$

and [2],

$$\mathcal{F}(l,p) = \sum_{i=l}^{p-1} \sum_{m=i+1}^{p} \not k_i \not k_m . \tag{15}$$

Forde & DAK (2005);

$A_n(3-)$: Berger, Bern, Dixon, Forde, & DAK (2006)

Summary

- The combination of the unitarity-based method and on-shell recursion relations gives a powerful and practical method for a wide variety of QCD calculations needed for LHC physics
- The same techniques are applicable to interesting questions in N =4 supersymmetry and in gravity
- Advances in gauge theory advance gravity via KLT & unitarity
- Lots of important calculations are now feasible, and are awaiting physicists eager to do them!