Classical and Quantum Strings in AdS/CFT

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Motivations

- There has been a good deal of success recently in comparing the energies of semi-classical string solutions with the anomalous dimensions of gauge invariant operators in the context of AdS/CFT.
- So far much of the progress has involved considering specific string solutions e.g. the solutions with certain large charges of Frolov, Tseytlin, GKP, BMN and many others
- In particular the possible presence of integrability in the classical worldsheet theory (Bena, Polchinski, Roiban also KMMZ and others) and in the dual gauge theory (Minahan&Zarembo) has led to the introduction of a number of powerful tools e.g. the Bethe ansatz.
- The S-matrix seems to be a particularly simple tool to describe the system and there has been a good deal of progress in finding the correct S-matrix for the gauge theory and worldsheet theory using symmetries, generalised crossing, perturbative results, wild conjectures,…(AFS, BDS, HL, FK,RSS, Janik, Beisert,Staudacher, HM…)
- Of course it would be useful to have a direct way to calculate and test the various conjectured S-matrices and in this talk I will try to outline a few partial results (and problems, presumably technical) with calculating the worldsheet S-matrix.
- Based on work in progress with T. Klose, R. Roiban, K. Zarembo

Outline

- Briefly outline Metsaev&Tseytlin Green-Schwarz string theory on supercoset space

$$\frac{\mathrm{PSU}(2,2|4)}{SO(4,1)\times SO(5)}$$

and in particular the construction of a gauge fixed light-cone action.

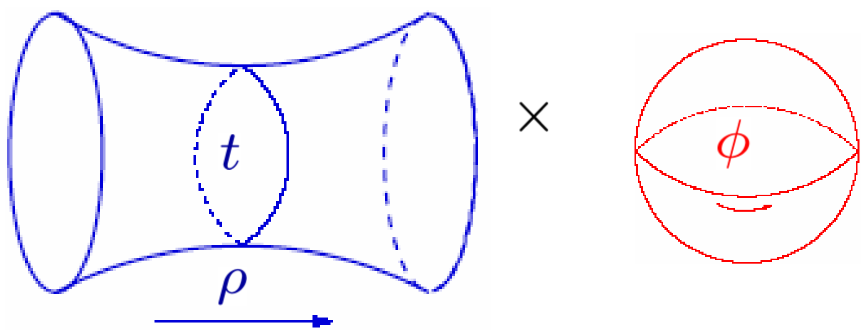
- Describe the calculation of classical string S-matrix in light-cone gauge near the BMN limit and demonstrate factorization for the rank one sector of a single complex boson.
- Find the one-loop corrections to four-point function for this action and in particular show that one runs across unexpected divergences.
- Consider the one-loop effective action of sigma model on supergroup manifold which is finite and describes a step toward finding one-loop effective action of the string theory in conformal gauge.

$AdS_5 \times S^5$ Geometry

- The metric in global coordinates is given by:

$$ds^2 = R^2(-\cosh^2\rho\, dt^2 + d\rho^2 + \sinh^2\rho\, d\Omega_3^2 + \cos^2\theta\, d\phi^2 + d\theta^2 + \sin^2\theta\, d\tilde{\Omega}_3^2$$

where R denotes the radius of both the AdS_5 and S^5 subspaces, and $d\Omega_3^2$, $d\tilde{\Omega}_3^2$ are 3-spheres. The coordinate ϕ has periodicity 2π.



- We must use the covering space so that t is not periodic
- Bosonic isometry group, SO(4,2) $\times$ SO(6), combines with the supersymmetries into the supergroup PSU(2,2|4)

- The string action in this background is

$$S = \frac{R^2}{4\pi} \int d^2\sigma \; (\sqrt{-h} h^{ab} G_{MN}(X) \partial_a X^M \partial_b X^N) + \text{fermions}$$

- Action is non-linear and quantization of string theory in this background is extremely difficult & so far unsolved.
- It is only tractable in certain limits when expanding about certain classical solutions; spinning string solutions (Frolov, Tseytlin,…2002-2003) the fast point-like string moving on a geodesic on the sphere (BMN).
- On a brighter note the action is integrable (Bena, Polchinski, Roiban) and so there are a number of powerful tools at ones disposal.

String S-matrix from scattering

- First we discuss expanding the Metsaev & Tseytlin action for the GS string in the near-Penrose limit and using the light-cone Lagrangian to find the S-matrix. The action is

$$\mathcal{S} = -\frac{1}{2}\int_{\partial M_3} d^2\sigma \, \sqrt{g} g^{ab} L^\mu_a L^\mu_b + i \int_{M_3} s^{IJ} L^\mu \wedge \bar{L}^I \Gamma^\mu \wedge L^J$$

- The L^μ and L^α are the bosonic and fermionic components of the super-vielbien
- $AdS_5 \times S^5$ can be written as a coset manifold which makes finding the vielbien possible (Kallosh, Rahmfeld, Rajaraman)

$$L^{\alpha J}_b = \frac{\sinh \mathcal{M}}{\mathcal{M}} \mathcal{D}_b \theta^{\alpha J}$$

$$L^\mu_a = e^\mu{}_\rho \partial_a x^\rho - 4i\bar{\theta}^I \Gamma^\mu \left(\frac{\sinh^2(\mathcal{M}/2)}{\mathcal{M}^2} \right) \mathcal{D}_a \theta^I$$

with

$$(\mathcal{D}_a\theta)^I = \left(\partial_a \theta + \frac{1}{4} \left(\omega^{\mu\nu}{}_m \, \partial_a x^m \right) \Gamma^{\mu\nu} \theta \right)^I - \frac{i}{2} \epsilon^{IJ} e^\mu{}_m \, \partial_a x^m \Gamma_* \Gamma^\mu \theta^J$$

$$(\mathcal{M}^2)^{IL} = \epsilon^{IJ} (\Gamma_* \Gamma^\mu \theta^J \bar{\theta}^L \Gamma^\mu) + \frac{1}{2} \epsilon^{KL} (-\Gamma^{jk} \theta^I \bar{\theta}^K \Gamma^{jk} \Gamma_* + \Gamma^{j'k'} \theta^I \bar{\theta}^K \Gamma^{j'k'} \Gamma'_*)$$

Worldsheet Action

- There are several issues involved in finding the appropriate action.

General Outline:

- Introduce light-cone coordinates and rescale the space-time fields.
- We expand the action in powers of 1/R, the radius of the AdS space (equiv. inverse powers of $\sqrt{\lambda}$).
- We fix $x^+ = p_- \tau$ and using kappa-symmetry $\Gamma^+ \theta = 0$
- Remove x^- using the constraint equations using the metric equations of motion.
- Determine the world sheet metric using the x^- equation of motion
- We calculate the light-cone Hamiltonian and express it in terms of the transverse coordinates & momenta

$$-P_+ \equiv H_{l.c.}(p^I, x'^I, x^I, \rho, \psi', c.c)$$

- At this point it is necessary to make a redefinition of the fermions in order to get canonical Poisson brackets.
- Legendre transform in the transverse directions to find the light-cone Lagrangian.

Alternatively

- Alternatively we can simply gauge fix the Lagrangian and Legendre transform in the remaining light-cone direction. This again gives an action for the transverse degrees of freedom.

$$\mathcal{L}_{\rm l.c.} = \mathcal{L}_{g.f.} - p_- \dot{x}^-$$

- This approach gives the same answer for the bosonic part of the action however the fermions may be different and it may be necessary to make a field redefinition to find agreement.
- Finally, for "constant J" gauge one can T-dualize in the ϕ direction and choose the gauge $t = \tau$, $\tilde{\phi} = \kappa\sigma$. We can now impose the appropriate κ-gauge (using the projector orthogonal to the fermion kinetic operator) and integrate out the world-sheet metric giving a Nambu-like action (similar to Kruczenski, Ryzhov and Tseytlin).

$$L = -\frac{1}{\kappa}\sqrt{-\det h} + L_{\rm WZ}$$

- We now expand in powers of fields and we find the same answer as the previous method provided we make the appropriate gauge choice and for a suitable definition of the fermions.

Light-cone gauge

This leads us to the issue of different gauge choices: two useful choices are

$$x^+ = t = p_- \tau \qquad x^- = R^2(\phi - t)$$

– where we choose $J=R^2 p_-$ to be constant and this gauge choice should give a s-matrix which agrees with the small momentum limit of AFS (once we take into account the difference in the definition of the string length).

– The "uniform" gauge, corresponding to E+J constant, the formula are a little simpler and the scattering matrix should agree with that of Frolov, Plefka and Zamaklar.

$$x^+ = \frac{t+\phi}{2} = p_- \tau \qquad x^- = R^2(\phi - t)$$

– Of course in the end these different choices should give equivalent physical descriptions.

Bosons at tree level

- Starting from the metric

$$ds^2 = R^2\left[-\left(\frac{1+\frac{1}{4}z^2}{1-\frac{1}{4}z^2}\right)^2 dt^2 + \left(\frac{1-\frac{1}{4}y^2}{1+\frac{1}{4}y^2}\right)^2 d\phi^2 + \frac{dz_k dz_k}{(1-\frac{1}{4}z^2)^2} + \frac{dy_{k'}dy_{k'}}{(1+\frac{1}{4}y^2)^2}\right]$$

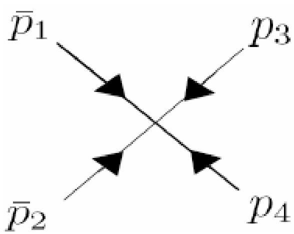
we introduce the light-cone fields and take the Penrose limit

$$t \to x^+ - \frac{x^-}{R^2} \qquad \phi \to x^+ + \frac{x^-}{R^2} \qquad z_k \to \frac{z_k}{R} \qquad y_{k'} \to \frac{y_{k'}}{R}$$

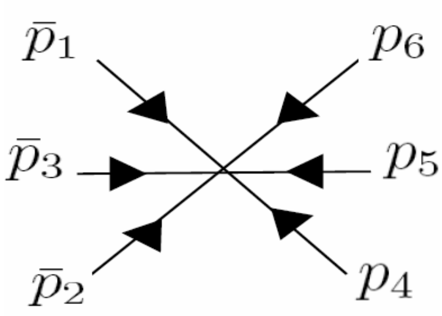
The light-cone Lagrangian restricted to fields on the sphere is given by

$$\begin{aligned}
\mathcal{L}_{\rm lc} &= p_y\dot{y} - \mathcal{H}_{\rm lc} \\
&= -\frac{1}{2}\left(-\dot{y}^2 + y'^2 + \frac{p_-^2}{2}y^2\right) + \frac{1}{2R^2}y^2y'^2 \\
&\quad + \frac{1}{32R^4p_-^2}\left(-2y^2y'^4 + p_-^2y^4\dot{y}^2 - 2y^2\dot{y}^4 - y'^2(9p_-^2y^4 + 4y^2\dot{y}^2) + 8y^2(\dot{y}\cdot y')^2\right) \\
&\quad + \frac{1}{8R^6p_-^2}\left(y^4y'^4 + y^4y'^2(p_-^2y^2 + \dot{y}^2) - 2y^4(\dot{y}\cdot y')^2\right) + O\left(\frac{1}{R^8}\right).
\end{aligned}$$

- This is simply a massive interacting scalar field theory and we can calculate the tree level scattering. If we further restrict our attention to two directions on the sphere y^5 & y^6 and we introduce the complex coordinate $y = y^5 + i\, y^6$ we can straightforwardly write down the Feynman rules for the vertices



$$\frac{-ip_-}{2J}(n_1 + n_2)(n_3 + n_4)$$



$$\frac{i}{32J^2}\left(8(\omega_1\omega_2 + \omega_1\omega_3 + \omega_2\omega_3 - (n_1n_2 + n_1n_3 + n_2n_3)) \times (\omega_4\omega_5 + \omega_4\omega_6 + \omega_5\omega_6 - (n_4n_5 + n_4n_6 + n_5n_6)) + 4p_-^2(\omega_1 + \omega_2 + \omega_3)(\omega_4 + \omega_5 + \omega_6) - 36p_-(n_1 + n_2 + n_3)(n_4 + n_5 + n_6)\right)$$

- From the four vertex we can read off the tree level scattering matrix. Using the relation

$$\delta(\omega_p+\omega_{p'}-\omega_k-\omega_{k'})\delta(n_p+n_{p'}-n_k-n_{k'}) \;=\;$$

$$\frac{\omega_p\omega_{p'}}{\omega_{p'}n_p-\omega_p n_{p'}}\left(\delta(n_p-n_k)\delta(n_{p'}-n_{k'})+\delta(n_p-n_{k'})\delta(n_{p'}-n_k)\right)$$

expressing the fact that in two dimensions particles can only exchange momenta and including the normalization of the mode creation/annihilation operators we find for the tree-level 2-particle S-matrix

$$S(n_p,n_{p'})=\frac{-i}{4J}\frac{(n_p+n_{p'})^2}{n_p\omega_{p'}-n_{p'}\omega_p}$$

This should agree with the small momentum limit of FPZ S-matrix and up to differences of notation it does so.

- We can carry out exactly the same procedure for the “uniform” gauge. The quartic term for the bosons on the sphere is

$$L_4 = \frac{1}{8p_-^2}(\partial_\sigma y^i + \partial_\tau y^i)^2(\partial_\sigma y^j - \partial_\tau y^j)^2 + \frac{1}{2}y^2(\partial_\sigma y)^2 - \frac{1}{8}p_-^2 y^4$$

from which we can extract the quartic vertex

$$V = \frac{1}{16p_-^2\sqrt{\omega_k\omega_{k'}\omega_p\omega_{p'}}} \times \left[2(n_k n_{k'} - \omega_k\omega'_k)(n_p n_{p'} - \omega_p\omega_{p'}) + 2p_-^2(n_p + n_{p'})(n_k + n_{k'}) - 2p_-^4)\right]$$

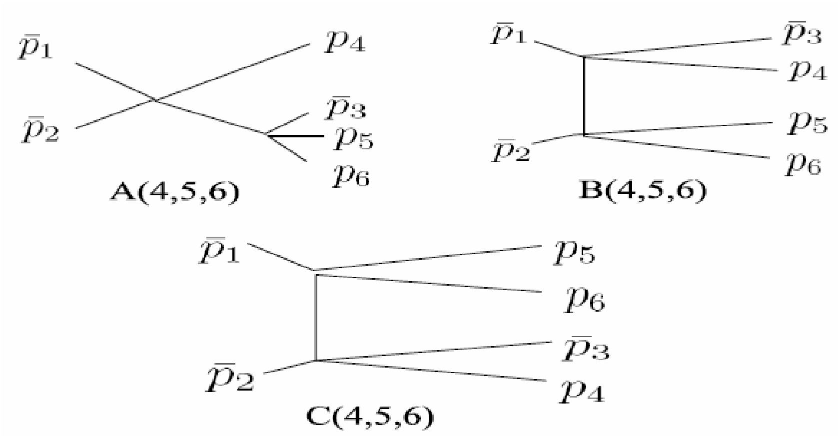
and including the appropriate kinematical factors we find

$$S(p,p') = (n_p - n_{p'}) + S_{RTT}^{su(2)}$$

where S_{RTT} is the S-matrix of Roiban, Tirziu and Tseytin corresponding to the small momentum limit of AFS and the extra piece corresponds to the difference in the definition of the string coordinate σ. There are similar results for the non-compact bosonic sl(2) sector.From now on we will focus on the “uniform“ gauge as the formula are simpler but anlogous results will hold in the “constant J” gauge.

Factorization

- We can now calculate the 2->4 scattering amplitude: for which we need the following diagrams



and their permutations. We further need the contribution of the six point vertex and summing all the contributions we find that they cancel. (Actually checked this is true for a range of numerical values and analytically in certain simplifying limits such as the momenta much larger than the masses). The absence of particle production is equivalent to factorization of the scattering matrix.

$$\mathbf{A}(4,5,6): \quad \frac{p_-^2}{2J^2}\left(\frac{(n_1+n_2)^2(n_5+n_6)^2}{(\omega_1+\omega_2-\omega_4)^2-(n_1+n_2-n_4)^2-p_-^2/2}\right)+(4\leftrightarrow 5)+(4\leftrightarrow 6)$$

$$\mathbf{B}(4,5,6): \quad \frac{p_-^2}{2J^2}\left(\frac{(n_1-n_3)^2(n_5+n_6)^2}{(\omega_1-\omega_3-\omega_4)^2-(n_1-n_3-n_4)^2-p_-^2/2}\right)+(4\leftrightarrow 5)+(4\leftrightarrow 6)$$

$$\mathbf{C}(4,5,6): \quad \frac{p_-^2}{2J^2}\left(\frac{(n_2-n_3)^2(n_5+n_6)^2}{(\omega_1-\omega_5-\omega_6)^2-(n_1-n_5-n_6)^2-p_-^2/2}\right)+(4\leftrightarrow 5)+(4\leftrightarrow 6)$$

$$\begin{aligned}\mathbf{D}: \quad & \frac{i}{32J^2}\left(8(\omega_1\omega_2-\omega_1\omega_3-\omega_2\omega_3-(n_1n_2-n_1n_3-n_2n_3))\times\right.\\ & \quad (\omega_4\omega_5+\omega_4\omega_6+\omega_5\omega_6-(n_4n_5+n_4n_6+n_5n_6))\\ & \left.-4p_-^2(\omega_1+\omega_2-\omega_3)(\omega_4+\omega_5+\omega_6)+36p_-^2(n_1+n_2-n_3)(n_4+n_5+n_6)\right)\end{aligned}$$

$$A(4,5,6)+B(4,5,6)+C(4,5,6)+(4\leftrightarrow 5)+(4\leftrightarrow 6)+D=0$$

as expected.

It would be good to extend this calculation to other sectors and in particular to include the fermions. For 2->4 fermions we would use the action

$$L_2 = \frac{-ip_-}{2}\left(\bar{S}\rho^\alpha \partial_\alpha S - ip_- \bar{S}\Pi S\right)$$

$$\begin{aligned} L_{4+6}^{fermions} \;&=\; \frac{ip_-}{2}\left[\bar{S}\rho^0\left(\frac{\mathcal{M}^2}{12}+\frac{\mathcal{M}^4}{360}\right)(\partial_\tau S - ip_-\Pi\rho^0 S)\right. \\ &\left. +\bar{S}\rho^1\left(\frac{\mathcal{M}^2}{12}+\frac{\mathcal{M}^4}{360}\right)\partial_\sigma S\right]. \end{aligned}$$

with $\mathcal{M}^2 = -\frac{1}{4}\rho^0\Pi\gamma^i S\bar{S}\gamma^i\rho^0 - \frac{1}{4}\rho^0\Pi\gamma^{i'} S\bar{S}\gamma^{i'}\rho^0 - \frac{1}{8}(\gamma^{ij}S\bar{S}\gamma^{ij} - \gamma^{i'j'}S\bar{S}\gamma^{i'j'})$

Quantum Corrections

- Now we wish to calculate the one loop corrections to the four point function and for this we will need the full Lagrangian incl. fermions and all the other bosons.
- The full light-cone bosonic Lagrangian to quartic order in fields is

$$L_2 = \frac{1}{2}\left[(\partial_\tau z)^2 - (\partial_\sigma z)^2 + (\partial_\tau y)^2 - (\partial_\sigma y)^2 - p_-^2(z^2+y^2)\right]$$

$$L_4 = \frac{1}{2}y^2(\partial_\sigma y)^2 - \frac{1}{2}z^2(\partial_\sigma z)^2 + \frac{1}{4}y^2\left[(\partial_\sigma z)^2 + (\partial_\tau z)^2\right] - \frac{1}{4}z^2\left[(\partial_\sigma y)^2 + (\partial_\tau y)^2\right]$$

while the terms quadratic in fermions are

$$L_2 = \frac{-ip_-}{2}\left(\bar{S}\rho^\alpha\partial_\alpha S - ip_-\bar{S}\Pi S\right)$$

$$\begin{aligned} L_4 = & -\frac{i}{8}(p_-(\bar{S}\rho^0\partial_\tau S - \bar{S}\rho^1\partial_\sigma S)y^2 + i\bar{S}\Pi S((\partial_\sigma y)^2 - (\partial_\tau y)^2)) \\ & -\frac{ip_-}{16}(\bar{S}\rho^0\Gamma^{ij}S(\partial_\tau y^i\, y^j) - 3\bar{S}\rho^1\Gamma^{ij}S(\partial_\sigma y^i\, y^j)) + \frac{1}{4}\bar{S}\rho^0\rho^1\Gamma^{ij}S\partial_\sigma y^i\partial_\tau y^j \end{aligned}$$

with the fermionic fields S being eight component Majoranna spinors and the ρ's are two dimensional Dirac matrices.

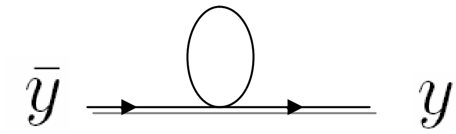
- For completeness here is the rest of the relevant Lagrangian:
  - terms with two AdS bosonic coordinates

$$
\begin{aligned}
L_6 \;=\; & -\frac{(\partial_\sigma y)^4 y^2}{32 p_-^2} - \frac{(\partial_\sigma y)^2(\partial_\tau y)^2 y^2}{16 p_-^2} - \frac{(\partial_\tau y)^4 y^2}{32 p_-^2} - \frac{(\partial_\sigma y)^2(\partial_\sigma z)^2 y^2}{16 p_-^2} - \frac{(\partial_\tau y)^2(\partial_\sigma z)^2 y^2}{16 p_-^2} \\
& -\frac{(\partial_\sigma y)^2(\partial_\tau z)^2 y^2}{16 p_-^2} - \frac{(\partial_\tau y)^2(\partial_\tau z)^2 y^2}{16 p_-^2} - \frac{(\partial_\sigma z)^4 y^2}{32 p_-^2} - \frac{(\partial_\sigma z)^2(\partial_\tau z)^2 y^2}{16 p_-^2} - \frac{(\partial_\tau z)^4 y^2}{32 p_-^2} \\
& -\frac{9(\partial_\sigma y)^2 y^4}{32} + \frac{(\partial_\tau y)^2 y^4}{32} - \frac{1}{16}(\partial_\sigma z)^2 y^4 + \frac{1}{16}(\partial_\tau z)^2 y^4 - \frac{(\partial_\sigma y)^4 z^2}{32 p_-^2} \\
& -\frac{(\partial_\sigma y)^2(\partial_\tau y)^2 z^2}{16 p_-^2} - \frac{(\partial_\tau y)^4 z^2}{32 p_-^2} + \frac{1}{4}(\partial_\sigma y)^2 y^2 z^2 + \frac{1}{32} p_-^2 y^4 z^2 \\
& +\frac{1}{8 p_-^2}(\partial_\sigma y \cdot \partial_\tau y)^2 (y^2 + z^2) + \frac{1}{4 p_-^2} y^2 (\partial_\sigma y \cdot \partial_\tau y)(\partial_\sigma z \cdot \partial_\tau z)
\end{aligned}
$$

  - relevant terms with two fermions and four bosons

$$
L_6 \;=\; \frac{i}{32 p_-}(\bar{S}\rho^0 \partial_\tau S - \bar{S}\rho^1 \partial_\sigma S) y^2 ((\partial_\tau y)^2 + (\partial_\sigma y)^2) - \frac{1}{32}\bar{S}\Pi S(4 y^2 (\partial_\sigma y)^2 - p_-^2 y^4)
$$

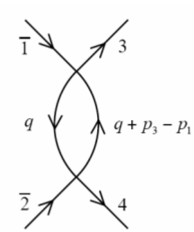
- We can now calculate the one-loop correction to the bosonic two point function and combining contributions from the bosons and fermions we find

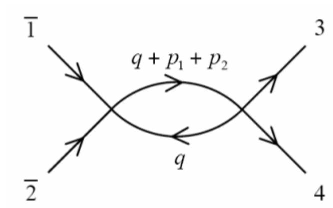


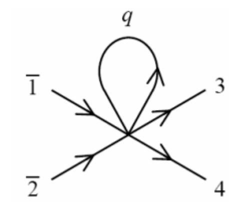
$$G^{(2)}(\bar{1},2) = (\omega_1\omega_2 - n_1 n_2 - p_-^2)\int \frac{d^{2-2\epsilon}q}{(2\pi)}\frac{1}{q^2+p_-^2} + finite$$

- If we now impose energy/momentum conservation and the quadratic equations of motion the divergent term goes away.
- This occurs as a non-trivial combination of the bosonic and fermionic contributions and after dropping integrals of the type $\int d^{2-2\epsilon}q \times 1$.
- Even off-shell such a divergence can be removed by a redefinition of the fields.

- We now calculate the four point function which receives contributions from terms involving
  - two quartic vertices and vertices with six fields







  - If we calculate in the c.o.m frame and use the dispersion relation we find that

$$G^{(4)}(p,-p,p',-p') \sim \frac{1}{p_-^2}(\omega^2 n_{p'}^2 + \omega_{p'}^2 n^2) \int \frac{d^{2-2\epsilon}q}{(2\pi)} \frac{1}{q^2+p_-^2} + finite$$

  - which does not vanish nor does it seem to be removable by renormalizing the fields.
- It is perhaps most illuminatingly written as a divergent contribution to the effective action in coordinate space and for the full SO(4) vectors (and after using the equations of motion to simplify the expressions)

$$\left[-\frac{1}{p_-^2}(\partial_\tau y \cdot \partial_\sigma y)^2 + \frac{1}{p_-^2}(\partial_\sigma y)^2(\partial_\tau y)^2 - 2y^2(\partial_\sigma y)^2\right] \int \frac{d^{2-2\epsilon}q}{q^2+p_-^2} + \text{finite}$$

- Several comments are in order
  - In light-cone gauge with a curved world-sheet metric the ghosts do not decouple and so we must include their contribution at one loop. There is no contribution to the two point function and unfortunately they do not remove the divergences from the four point function.
  - The finite part of the four point function does not vanish when one considers zero-mode states, these excitations are BPS and so should remain free. A similar problem arises in the calculation of near-BMN energies and is remedied by making a field redefinition for the fermions. In fact we are allowed to make arbitrary field redefinitions e.g.

$$S \to S + M(y)S, \qquad M(y) = A(y) + \rho^0 B(y) + \rho^1 C(y) + \rho^0 \rho^1 D(y)$$

    and we expect that a similar redefinition will remove the zero-mode interactions in this case. We can further ask if a field redefinition will remove the divergences unfortunately this does not seem to be the case. However it is possible that a more general redefinition may work
  - There is also the issue of world-sheet diffeomorphism invariance. We might expect that a different choice of gauge would remove these divergences. Certainly trying a different light-cone gauge where J is uniformly distributed rather than P_- does not seem to fix these issues however perhaps there is a more general transformation that would.

Sigma model on supergroup manifold

Given these difficulties it is perhaps useful to consider the string action in conformal gauge. As a toy model we examine the sigma model on the supergroup manifold psu(2|2) a la Bershadsky et al which is related to the super string in/on Euclidean AdS_3 x S^3 . This theory contains a bosonic su(2) and we could hope that many of the features will be shared by strings on S^3 indeed we could further hope that many features would be relevant to strings on the psu(2,2|4) supercoset manifold.

Two dimensional non-linear sigma model with the fields, G(x), taking values in supergroup PSU(2|2). The Lie algegbra of PSU(2|2) consists of the 4 x 4 supermatrices (bosonic diagonal blocks and fermionic off diagonal blocks), with vanishing supertrace and which satisfy a specific reality condition (super antihermiticity).

$$M = \begin{pmatrix} A & X \\ Y & B \end{pmatrix} \quad \text{where} \quad trA = trB = 0 \quad \& \quad M^\dagger \equiv \begin{pmatrix} A^\dagger & -iY^\dagger \\ -iX^\dagger & B^\dagger \end{pmatrix} = -M$$

Sigma model on supergroup manifold

We will consider the action $$S = \frac{-1}{4\pi\lambda^2}\int d^2z\,\eta^{\mu\nu}\mathrm{STr}[\mathrm{J}_\mu \mathrm{J}_\nu]$$

where the $J = G^{-1}dG$ are super-Lie algebra valued currents. Now we want to find the four field contribution to the 1PI effective action. We choose an arbitrary background, G_0 , which we demand satisfy the equations of motion $\partial_\mu J_0^\mu = 0$.

- We expand the action in powers of fluctuations, that is we choose $G = G_0 P \quad \text{with} \quad P = e^{\lambda X}$ expand in λ and we find

$$\begin{aligned} S \;=\; \frac{-1}{4\pi\lambda^2}\int d^2z\,\eta^{\mu\nu}\Big[\; & \mathrm{STr}[J_\mu^{(0)} J_\nu^{(0)}] \\ & + \mathrm{STr}[\,\lambda^2\partial_\mu X\partial_\nu X - \frac{1}{12}\lambda^4[\partial_\mu X,\, X][\partial_\nu X,\, X]\;] + \ldots \\ & + \lambda^2\;\mathrm{STr}[J_\mu^{(0)}[\partial_\nu X,\, X]] + \frac{1}{3}\lambda^3\;\mathrm{STr}[[J_\mu^{(0)},\, X][\partial_\nu X,\, X]] + \ldots \end{aligned}$$

- We now simply integrate out the fluctuation fields and resulting effective action will generate the quantum s-matrix.

- It is useful to review a few facts about the supergroup invariant tensors. We choose the generators to be in the fundamental representation so that the quantity

$$\gamma_{ab}^{R} = \ \mathrm{STr}_R[T_a T_b]$$

gives an invariant non-degenerate metric. For arbitrary representations the structure constants

$$f_{abc}^{(R)} = \ \mathrm{STr}[[T_a,\, T_b]\, T_c] = f_{ab}{}^{d} \gamma_{dc}^{(R)}$$

satisfy the following relations

$$(-1)^{[m]} \gamma_R^{mn} f_{nap} \gamma_R^{pq} f_{qbm} = 0$$

$$(-1)^{[m]} \gamma_R^{mn} f_{nap} \gamma_R^{pq} f_{qbs} \gamma_R^{st} f_{tcm} = 0$$

which corresponds to the vanishing of the dual Coxeter number and the triple identity. Further, at least for the fundamental representation, we have the following quartic identity:

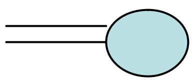
$$(-1)^{[m]} \gamma^{mn} f_{nap} \gamma^{pq} f_{qbs} \gamma^{st} f_{tcu} \gamma^{uv} f_{vdm} = 2(\gamma_{ab}\gamma_{cd} + \gamma_{ac}\gamma_{bd} + \gamma_{ad}\gamma_{bc})$$

- The fluctuations are massless relativistic scalars with the usual kinetic term and propagator

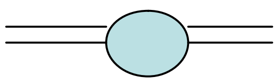
$$\Delta^{ab}(z_1 - z_2) = \int \frac{d^2p}{2\pi} \frac{e^{ip\cdot(z_1-z_2)}}{2(-p^2)} \gamma^{ba}$$

- To calculate the one loop diagrams we only need a single cubic vertex:

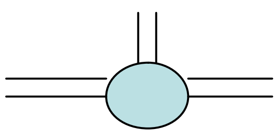
$$-iV = -\frac{1}{2} \int \frac{\prod d^2p_i}{(2\pi)^2} \delta(\Sigma p_i) \tilde{J}^a(p_1) \cdot p_2 X^c(p_2) X^d(p_3) \gamma_{ab} f^b_{cd}$$



$$\propto \gamma^{dc} f_{cd}{}^{b}$$



$$\propto (-1)^{[m]} \gamma^{mn} f_{nap} \gamma^{pq} f_{qbm} \propto C_V = 0$$



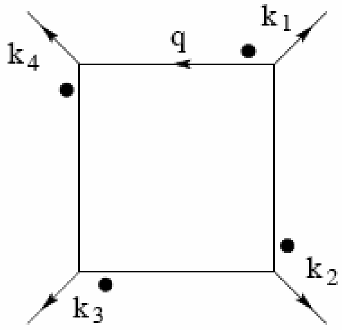
$$\propto (-1)^{[m]} \gamma^{mn} f_{nap} \gamma^{pq} f_{qbs} \gamma^{st} f_{tcm} = 0$$

- Thus the one, two and three point functions all vanish at one loop (as was already known (in fact they vanish to all orders in perturbation theory)).

- The first non-trivial contribution to the one-loop effective action

$$\frac{1}{(4\pi)}\int\frac{\prod d^2k_i}{(2\pi)}\delta(\Sigma k_i)L_{eff}^{1-loop}(k_1,k_2,k_3,k_4)$$

is the four field term coming from the graph



$$-\frac{1}{4}(-)^{[c1]}\gamma^{c_1a_4}f_{a_4b_4c_4}\gamma^{c_4a_3}f_{a_3b_3c_3}\gamma^{c_3a_2}f_{a_2b_2c_2}\gamma^{c_2a_1}f_{a_1b_1c_1}$$
$$J_{(0)}^{\rho_4\,b_4}(k_4)J_{(0)}^{\rho_3\,b_3}(k_3)J_{(0)}^{\rho_2\,b_2}(k_2)J_{(0)}^{\rho_1\,b_1}(k_1)I_{\rho_4\rho_3\rho_2\rho_1}(k_4,k_3,k_4,k_1)$$

where the loop integral is

$$I_{\rho_4\rho_3\rho_2\rho_1}(k_4,k_3,k_4,k_1)=\int\frac{d^2q}{(2\pi)^2}\frac{(q+k_{123})_{\rho_4}(q+k_{12})_{\rho_3}(q+k_1)_{\rho_2}q_{\rho_1}}{(q+k_{123})^2(q+k_{12})^2(q+k_1)^2q^2}$$

Fortunately there are simple ways to evaluate two dimensional loop integrals of this sort (Källén & Toll ‘64).

- We introduce l.c. momenta $q_s = q_0 + q_1$ and $q_t = q_0 - q_1$ and the non-vanishing integrals are

$$F = I_{ttss}(p_4, p_3, p_2, p_1)$$

and various permutations where

$$p_1 = 0 \; ; p_2 = -k_1 \quad ; p_3 = -(k_1 + k_2) \; ; p_4 = -(k_1 + k_2 + k_3)$$

We now shift the poles off the real line

$$F = \frac{1}{2(2\pi)} \int dq_s \; dq_t \frac{(q_s - p_{1,s})(q_s - p_{2,s})(q_t - p_{3,t})(q_t - p_{4,t})}{\prod_i (q_t - p_{i,t})(q_s - p_{i,t}) + i\epsilon}$$

and we can perform the q_s integral by closing the contour in the lower half plane resulting in

$$F = \frac{i}{4} \int dq_t \frac{1}{(q_t - p_{1,t})(q_t - p_{2,t})(p_{3,s} - p_{4,s})} [\Theta(q_t - p_{3,t}) - \Theta(q_t - p_{4,t})]$$

which corresponds to integrating q_t over a finite region giving a final answer

$$F = \frac{i}{8} \ln \left(\frac{(\mathrm{p}_1 - \mathrm{p}_4)^2}{(\mathrm{p}_1 - \mathrm{p}_3)^2} \frac{(\mathrm{p}_2 - \mathrm{p}_3)^2}{(\mathrm{p}_2 - \mathrm{p}_4)^2} \right) \frac{1}{(\mathrm{p}_{3,\mathrm{s}} - \mathrm{p}_{4,\mathrm{s}})(\mathrm{p}_{1,\mathrm{t}} - \mathrm{p}_{2,\mathrm{t}})}.$$

- At this point we have evaluated the full one-loop four point contribution to the effective action. At the moment the expression is still a little complicated but it’s finite.
- We can similarly perform the two loop calculation; we now have several more graphs but again the contribution seems to be finite and $\propto f_{abr}\gamma^{rs}f_{cds}$ or $f_{acr}\gamma^{rs}f_{bds}$. At this point it is hard to draw too many conclusions from this but one notable feature is that the answer is always in terms logarithms of momenta.
- It is straightforward to do the same calculation for the coset sigma model and there are only a few extra diagrams which should also be finite.
- One could possibly include ghosts following Berkovits, Vallilo (pure spinor superstring is conformally invariant at loop level).

- Returning to the sigma model on the supergroup we can now expand about some classical solution and calculate an s-matrix. Two cases:
  - Choose a bosonic solution with fields in the su(2) of S^3 similar to that of the l.c. string theory. One important difference is that now we can't have a non-trivial metric so we must allow

$$x^+ = \frac{t+\phi}{2} = \kappa\tau + corr$$

where the correction piece is chosen so that the x^- equation of motion is satisfied. If we expand in powers of fields we again find a massive scalar field theory with a classical interaction given by

$$-\frac{y^2}{2}\left(-(\partial_\tau y)^2 + (\partial_\sigma y)^2 - \kappa y^2\right)$$

To find the one-loop correction we now need the full effective action and not merely the four term part as we will get contributions to the term quartic in y's and indeed to the mass term which involve all orders of κ. Otherwise we must make the further approximation $\kappa \ll p$ where p is momenta of excitations. (Not clear whether this is a consistent choice).

- A second choice is to expand about the trivial vacuum: we now have a conformal theory with massless fields and we find a s-matrix for the longitudinal and transverse fields. In this case there is the usual difficulty of defining what we mean by a s-matrix however one possibilty is to add a small mass, calculate the s-matrix and take the massless limit.
- This may give sensible results and we can interpret the S-matrix as the phase picked-up as two particles pass each other, however it is unclear how one would then use this.

Summary

- It is of interest to find as much information about the string s-matrix as possible and here we have described some partial results by two different methods.
  - Studying quantum corrections to the scattering of excitations in the near-BMN limit: tree level results seem to make sense (as expected) but at one loop we find divergences which are difficult to interpret.
  - Calculating the quantum effective action for supercoset sigma model. Here we find finite results which may give us hope that a similar approach for the full string theory may work.
  - Future work:
    - Extend classical results to all string sectors.
    - Find a redefinition to remove divergences in l.c. gauge or find a sensible interpretation.
    - Extract sensible s-matrix from supercoset calculation and extend results to full string theory (incl ghosts).