Four-point functions in the AdS/CFT correspondence: Old and new facts

Emery Sokatchev

Laboratoire d'Annecy-le-Vieux de Physique Théorique LAPTH Annecy, France

I. The early AdS/CFT correspondence

The AdS/CFT conjecture, in its weakest form, predicts the correspondence between KK states in Type IIB SG on $\text{AdS}_5 \times S^5$ and 1/2 BPS short gauge invariant operators in the boundary SCFT_4 of $\mathcal{N} = 4$ super-Yang-Mills (SYM) in the 't Hooft limit.

Gauge theory

$\mathcal{N} = 4$ SYM contains the fields

$$\phi^i, \; A_\mu, \; \psi^r_\alpha$$

$i = 1, \ldots, 6$ and $r = 1, \ldots, 4$ with action

$$\begin{aligned} S &= \frac{1}{g^2} \int d^4x \mathrm{Tr} \Big\{ \frac{1}{4} F^2_{\mu\nu} + \frac{1}{2} (D_\mu \phi^i)^2 + \bar{\psi} \gamma^\mu D_\mu \psi \\ &- \frac{1}{4} \sum_{ij} [\phi^i, \phi^j]^2 - \frac{i}{2} \Gamma_i [\bar{\psi}, \phi^i] \psi \Big\} \end{aligned}$$

This theory has a vanishing beta function and is thus (super)conformal.

One can prepare various gauge invariant composites, in particular, the so-called 1/2-BPS multiplets

$$\mathcal{O}_{[0k0]}(x) \sim \mathrm{Tr}(\phi^{(i_1}(x) \cdots \phi^{i_k)}(x))$$

They transform in the $[0k0]$ of the the R-symmetry group $SU(4)$ and are annihilated by half of the supercharges. Superconformal symmetry fixes their dimension $\Delta = k$.

Example: $[020]$ is the stress-tensor multiplet

$$\mathcal{O}_{20'}(x) \quad \stackrel{\mathcal{N}=4\ SUSY}{\Longrightarrow} \quad T_{\mu\nu} \sim \mathrm{Tr}(\partial_\mu \phi^i \partial_\nu \phi^i) + \dots$$

AdS supergravity

Type IIB supergravity in $D = 10$ contains the fields

$$(g_{MN}, B_{MN}, \phi | C_{MN}, A_{MNKL}, \chi)$$

plus fermions.

Compactification on S^5 results in an infinite tower of Kaluza-Klein (KK) modes K.

Duality

All KK states of this type IIB compactification are in correspondence (duality) with the 1/2 BPS operators of $\mathcal{N} = 4$ SYM

$$\mathcal{O}(x_i) \Leftrightarrow K(x_i, x_0)$$

AdS/CFT predicts matching correlators of dual states/operators:

$$\langle \mathcal{O}_1(x_1) \ldots \mathcal{O}_n(x_n) \rangle = \frac{\delta}{\delta K_1(x_1)} \cdots \frac{\delta}{\delta K_n(x_n)} S_{SG}(K_1, \ldots, K_n)$$

What can we test by comparing gauge amplitudes to their gravity counterparts?

Problem: Two different perturbative expansions in

$$\lambda = g_{YM}^2 N \quad \text{and} \quad \frac{\alpha'}{R^2} = \lambda^{-\frac{1}{2}}$$

$$\langle \mathcal{O} \ldots \mathcal{O} \rangle_{gauge} = \frac{1}{N^2} f_1(\lambda) + \frac{1}{N^4} f_2(\lambda) + \ldots$$

$$\langle \mathcal{O} \ldots \mathcal{O} \rangle_{string} = \frac{1}{N^2} h_1(\lambda^{-\frac{1}{2}}) + \frac{1}{N^4} h_2(\lambda^{-\frac{1}{2}}) + \ldots$$

Field theory expansion is around $\lambda = 0$ while string expansion is around $\lambda = \infty$. Semi-classical supergravity: $\lambda = \infty$ – inaccessible in filed theory. Still, we can test a few things:

– Three-point functions of 1/2 BPS operators are fixed by conformal SUSY up to dynamically determined normalization $C(g^2)$. AdS calculations (Lie, Minwalla, Rangamani, Seiberg) show that

$$C(g^2) = C(g = 0)$$

Non-renormalization theorem on the CFT side (D'Hoker, Freedman, Skiba; Howe, ES, West).

– Four-point functions of 1/2 BPS operators have functional freedom, e.g., for the stress-tensor multiplet

$$\langle \mathcal{O}^{I_1}_{20'}(x_1) \ldots \mathcal{O}^{I_4}_{20'}(x_4) \rangle = a(s,t) \frac{\delta^{I_1 I_2} \delta^{I_3 I_4}}{(x_{12}^2 x_{34}^2)^2} + b(s,t) \frac{C^{I_1 I_3 I_2 I_4}}{x_{13}^2 x_{14}^2 x_{23}^2 x_{24}^2} + \text{cyclic permutations of indices (2,3,4)}\,.$$

Here s, t are the conformal cross-ratios

$$s = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}\,, \qquad t = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}\,.$$

Partial non-renormalization theorem (Eden, Petkou, Schubert, ES):

$$a(s,t) = b(s,t)$$

This property is a manifestation of CFT dynamics.

On the AdS side one can compute the dual four-point amplitude for massless ($k = 2$) KK modes (Arutyunov, Frolov)

$$\langle K_2 K_2 K_2 K_2 \rangle$$

It has exactly the partially non-renormalized form predicted and observed in CFT.

What else can we learn from four-point correlators? Operator product expansion (OPE):

$$\langle \mathcal{O}(1) \ldots \mathcal{O}(4) \rangle$$
$$= \sum_{\ell,\Delta} \int_{5,6} \langle \mathcal{O}(1)\mathcal{O}(2)\mathcal{S}_{\ell,\Delta}(5) \rangle$$
$$\langle \mathcal{S}_{\ell,\Delta}(5)\mathcal{S}_{\ell,\Delta}(6) \rangle^{-1} \langle \mathcal{S}_{\ell,\Delta}(6)\mathcal{O}(3)\mathcal{O}(4) \rangle$$

– OPE lead to the discovery of semishort operators (Arutyunov, Frolov, Petkou) whose protection is explained by CSUSY kinematics (Arutyunov, Eden, ES)

– OPE gives information about the spectrum of long twist-two operators, e.g.

$$\mathcal{O}_j \sim \mathrm{Tr}\left(\phi^i \partial^{(\mu_1} \ldots \partial^{\mu_j)} \phi^i \right)$$

in CFT. They are absent (too heavy) in AdS.

No direct quantitative AdS/CFT comparison seems possible because of the different perturbative regimes.

II. Modern AdS/CFT spectroscopy. Search for integrability

Berenstein, Maldacena and Nastase (BMN) made an important step forward by proposing to study a particular class of operators in SCFT.

Make two complex combinations

$$X = \phi_1 + i\phi_2\,, \qquad Z = \phi_3 + i\phi_4$$

of the scalars ϕ^i ($i = 1, \ldots, 6$) of $\mathcal{N} = 4$ SYM (Z has a charge under $U(1) \subset SU(4)$). Form composites with many Z but few X ("impurities"), e.g.

BMN ops: $$\mathcal{O}_J \sim \sum_{k=0}^{J+2} c_k \mathrm{Tr}(X Z^k X Z^{J+2-k})$$

This operator has charge $J + 2$ and canonical dimension $\Delta = J + 4$. Without the impurities the operator is protected (no anomalous dimension); the simplest example of an operator with anomalous dimension is $J = 0$ (a member of the Konishi multiplet).

BMN correspondence: consider the double-scaling limit

$$J \rightarrow \infty, \quad N \rightarrow \infty, \quad \frac{J^2}{N} \rightarrow \text{fixed}$$

The BMN operators are dual to string states in a plane wave background. Their scaling dimension

$$\Delta = (J+4) + \gamma$$

should match the string energy

$$E = (J+2)+2\sqrt{1+\lambda'}+O(\frac{1}{J}) = (J+4)+\lambda'+\dots$$

where

$$\text{BMN coupling:} \qquad \lambda' = \frac{\lambda}{J^2} = \frac{g_{YM}^2 N}{J^2}$$

BMN scaling: the anomalous dimension γ is analytic around $\lambda' = 0 \Rightarrow$ admits perturbative expansion in g_{YM}^2. For the first time it became possible to compare perturbative CFT and string theory. BMN scaling has been verified at one loop (by BMN) and two loops (Gross et al).

Beisert, Kristjansen and Staudacher developed an algebraic way for determining anomalous dimension using the so-called dilatation operator (DO). This is a differential operator (wrt the fields). Applied to a set of bare composites, it produces a matrix whose eigenvalues are the anomalous dimensions.

In the X, Z (or $SU(2)$) sector the two-loop DO is fixed from symmetries (Beisert). At three loops there remains a two-parameter freedom. It can be fixed by the additional assumption of compatibility with BMN scaling. A direct three-loop computation (Eden, Jarczak, ES) for the two simplest operators from the BMN family ($J = 0, 1$) allowed to unambiguously determine the DO and thus to indirectly confirm BMN scaling.

The question is presently open whether BMN scaling breaks down at four loops?

The DO in $\mathcal{N} = 4$ SYM has the remarkable property of integrability. Minahan and Zarembo observed that the one-loop DO can be identified with the Heisenberg spin chain Hamiltonian known to describe an integrable system with an infinite set of conserved charges (a similar phenomenon in QCD has been known since the pioneering work of Lipatov). This crucial observation lead to the application of the powerful Bethe Ansatz technique for finding the BMN spectrum.

Recently, Rej, Serban and Staudacher made an all-loop proposal, the Hubbard model, compatible with BMN scaling. This model predicts, e.g., the four-loop anomalous dimensions of the operators with $J = 0, 1$ - to be tested!

III. Integrability and transcedentality

Kotikov, Lipatov, Onischenko and Velizhanin came up with a remarkable conjecture about the three-loop anomalous dimensions of the twist-two operators. They extracted it from the monumental QCD calculation of Moch, Vermaseren, Vogt by simply keeping the 'most complicated', i.e. the terms of maximal transcedentality in the form of multiple harmonic sums:

$$\begin{aligned}\gamma^{(3)}(j) &= 2\,\overline{S}_{-3}\,S_2 - S_5 - 2\,\overline{S}_{-2}\,S_3 - 3\,\overline{S}_{-5} + 24\,\overline{S}_{-2,1,1,1} \\ &\quad +6\Big(\overline{S}_{-4,1} + \overline{S}_{-3,2} + \overline{S}_{-2,3}\Big) - 12\Big(\overline{S}_{-3,1,1} + \overline{S}_{-2,1,2} + \overline{S}_{-2,2,1}\Big) \\ &\quad -\Big(S_2 + 2\,S_1^2\Big)\Big(3\,\overline{S}_{-3} + S_3 - 2\,\overline{S}_{-2,1}\Big) - S_1\Big(8\,\overline{S}_{-4} + \overline{S}^2_{-2} \\ &\quad +4\,S_2\,\overline{S}_{-2} + 2\,S_2^2 + 3\,S_4 - 12\,\overline{S}_{-3,1} - 10\,\overline{S}_{-2,2} + 16\,\overline{S}_{-2,1,1}\Big)\end{aligned}$$

and $S_a \equiv S_a(j)$, $S_{a,b} \equiv S_{a,b}(j)$, $S_{a,b,c} \equiv S_{a,b,c}(j)$ are harmonic sums

$$\begin{aligned} S_a(j) &= \sum_{m=1}^{j} \frac{1}{m^a}, \quad S_{a,b,c,\cdots}(j) = \sum_{m=1}^{j} \frac{1}{m^a}\, S_{b,c,\cdots}(m), \qquad (1) \\ S_{-a}(j) &= \sum_{m=1}^{j} \frac{(-1)^m}{m^a}, \quad S_{-a,b,c,\cdots}(j) = \sum_{m=1}^{j} \frac{(-1)^m}{m^a}\, S_{b,c,\cdots}(m), \\ \overline{S}_{-a,b,c,\cdots}(j) &= (-1)^j\, S_{-a,b,c,\ldots}(j) + S_{-a,b,c,\cdots}(\infty)\Big(1 - (-1)^j\Big). \end{aligned}$$

Soon afterwards, Staudacher was able to confirm this result from a completely different point of view, that of the factorized S-matrix, by using a perturbative asymptotic Bethe Ansatz.

In a parallel development, Bern, Dixon, Kosower, Smirnov studied four-gluon scattering amplitudes in $\mathcal{N} = 4$ SYM. The gluon legs are put on the massless shell. The resulting infrared and collinear singularities contain information about the asymptotic limit

$$\lim_{j\to\infty} \gamma^{(3)}(j) \sim \ln j$$

which exactly matches the KLOV prediction.

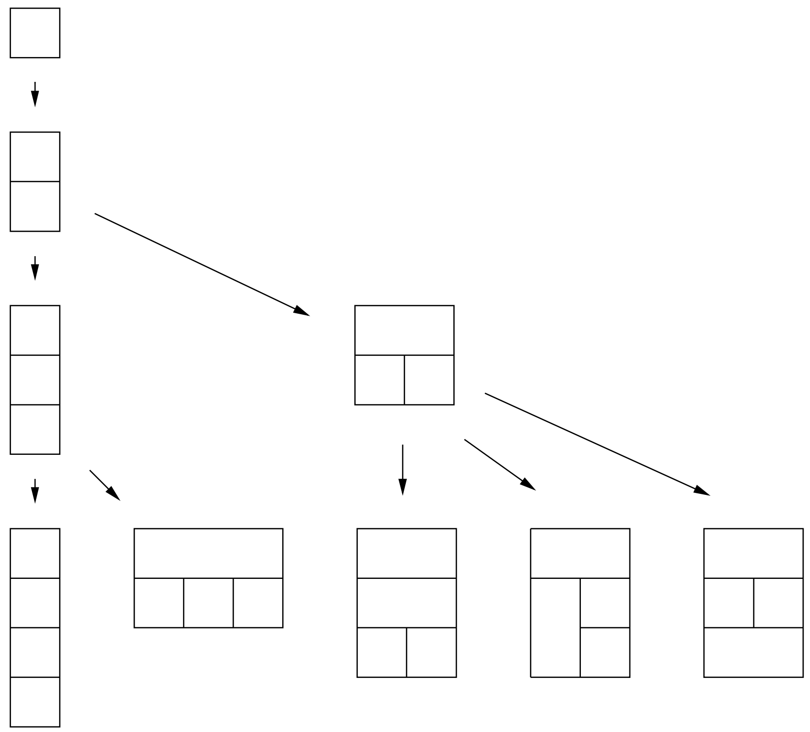
Another confirmation of the KLOV formula came from our calculation of $\gamma^{(3)}(j = 0)$ (Konishi). So, now we know $\gamma^{(3)}(j = 0)$ and $\gamma^{(3)}(j = \infty)$, but what happens in between? It is very important to prove this formula - maximal transcedentality is likely to be another manifestation of integrability (see recent work by Eden and Staudacher).

How could we do this?

– Direct calculation of $\gamma^{(3)}(j)$ by QCD methods - seems too difficult ($\mathcal{N} = 4$ SYM is more complicated than QCD)

– OPE of $\langle \mathcal{O}_{20'}(1) \ldots \mathcal{O}_{20'}(4) \rangle$. This has been very successful at one and two loops (Eden, Schubert, ES; Bianchi, Kovacs, Rossi, Stanev; Dolan, Osborn). The three-loop calculation is difficult but not impossible (under way?).

Alternatively, one may try to guess $\langle \mathcal{O}_{20'}(1) \ldots \mathcal{O}_{20'}(4) \rangle_{3\ loop}$ by putting together conformal four-point integrals. The OPE consistency conditions and crossing symmetry impose strong restriction, so one might hope to fix the possible form of the correlator. One and two loops seem to suggest a simple pattern - only the so-called ladder (scalar box) integrals appear. Unfortunately, at three loops the number of possible conformal integrals grows very rapidly (we know at least 26 integrals), and not all of them can be easily calculated.



Another approach would be to compose the correlator from harmonic polylogs of maximal transcedentality. Two independent attempts (Drummond, Smirnov, ES; Dolan, Heslop, Osborn) have failed so far - it is easy to meet the OPE requirements, but the KLOV formula is not reproduced.

We need to know the truth!