Integrability, Transcendentality, and the AdS/CFT Correspondence

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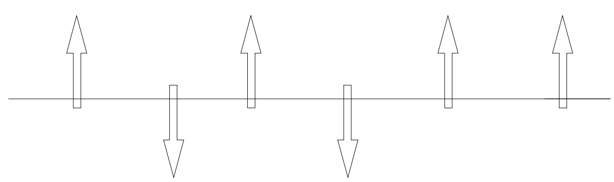


$\mathcal{N} = 4$ Super Yang-Mills Theory, I

Quantum Field Theory with gauge symmetry $SU(N)$. Fields:

The covariant gauge field A_μ, with $\mu = 1, \ldots, 4$, encoded into the invariant field strength $\mathcal{F}_{\mu\nu}$.

Six real scalars Φ_m with $m = 1, \ldots, 6$. In complex notation: $\mathcal{X} = \Phi_1 + i\,\Phi_2$, $\mathcal{Y} = \Phi_3 + i\,\Phi_4$, $\mathcal{Z} = \Phi_5 + i\,\Phi_6$ and $\bar{\mathcal{X}}, \bar{\mathcal{Y}}, \bar{\mathcal{Z}}$.

Four four-component fermions $\Psi^A_\alpha, \Psi^A_{\dot\alpha}$ with $A = 1, \ldots, 4$ and $\alpha, \dot\alpha = 1, 2$.

Lastly, infinitely many further fields are obtained by acting with covariant derivatives $\mathcal{D}_\mu = \partial_\mu + i\,A_\mu$ on the above. In complex notation: $\mathcal{D} = \mathcal{D}_1 + i\,\mathcal{D}_2$, $\mathcal{D}' = \mathcal{D}_3 + i\,\mathcal{D}_4$ and $\bar{\mathcal{D}}, \bar{\mathcal{D}}'$.

All fields are massless and adjoint, i.e. $N \times N$ matrices.

$\mathcal{N} = 4$ Super Yang-Mills Theory, II

The action:

$$S_{\mathcal{N}=4} = \frac{2N}{\lambda} \int d^4x \ \mathrm{Tr} \left(\tfrac{1}{4}(\mathcal{F}_{\mu\nu})^2 + \tfrac{1}{2}(\mathcal{D}_\mu \Phi_m)^2 - \tfrac{1}{4}[\Phi_m, \Phi_n]^2 + \ldots \right).$$

The action is unique due to the maximal number of $\mathcal{N} = 4$ supersymmetries, i.e. it does not renormalize.

The only adjustable parameters are

- The coupling constant g_{YM}, written in form of the 't Hooft coupling $\lambda = N\, g_{\mathrm{YM}}^2$. Parameter, does not "run".
- The number of colors N of the gauge algebra $\mathfrak{su}(N)$.
- A topological angle θ. Believed irrelevant in the planar $N \to \infty$ theory.

However, there is *wavefunction renormalization*, see below.

$\mathcal{N} = 4$ Super Yang-Mills Theory, III

In perturbation theory $\mathcal{N} = 4$ is a close relative of QCD.

Many Feynman diagrams are identical!

Extremely similar structures for gluon amplitudes and anomalous operator scaling dimensions.

Physically the theory is very different from QCD. It is quantum mechanically exactly scale-invariant, and thus conformal:

The theory is "finite" and the beta function vanishes:

$$\boxed{\beta = 0.}$$

Conformality combines with supersymmetry to superconformal symmetry $\mathfrak{psu}(2,2|4)$.

$\mathcal{N} = 4$ Super Yang-Mills Theory, IV

Conformal invariance strongly restricts the structure of correlation functions. For e.g. two-point functions:

$$\langle \mathcal{O}_n(x)\, \mathcal{O}_m(0) \rangle = \frac{\delta_{nm}}{x^{2\Delta_n}}\,.$$

Here Δ_n is the anomalous scaling dimension of the operator $\mathcal{O}_n$. "Good" operators are, in a complicated way, composed of the elementary fields. This leads to the mixing problem of $\mathcal{N} = 4$:

$$\mathcal{O} = \mathrm{Tr}\left(\mathcal{X}\mathcal{Y}\mathcal{Z}\mathcal{F}_{\mu\nu}\Psi^A_\alpha(\mathcal{D}_\mu\mathcal{Z})\ldots\right)\mathrm{Tr}\,(\ldots\ldots)\ \ldots + \ldots$$

The cleanest way to formulate and solve the mixing problem leads to an eigenvalue problem for a linear operator: The dilatation operator D :

$$D \cdot \mathcal{O}_n = \Delta_n\, \mathcal{O}_n\,.$$

Twist Operators in $\mathcal{N} = 4$ and Spin Chains, I

So anomalous scaling dimensions in a conformal gauge theory such as $\mathcal{N} = 4$ are defined as the eigenvalues of a linear operator, the dilatation operator D :

$$\boxed{D \cdot \mathcal{O} = \Delta\, \mathcal{O}\,.}$$

An important special case of this mixing problem are the model's twist operators:

$$\mathcal{O} = \mathrm{Tr}\left(\mathcal{D}^s \mathcal{Z}^J\right) + \dots\,.$$

Here $\mathcal{D} = \mathcal{D}_1 + i\,\mathcal{D}_2$ is a light-cone covariant derivative, where $\mathcal{D}_\mu = \partial_\mu + i\,A_\mu$, and s is the space-time spin.

The dilatation operator is related to the Hamiltonian of a quantum "spin chain". In the present example length=twist=J and the states are

$$\not{\mathrm{Tr}}\left((\mathcal{D}^{s_1}\mathcal{Z})(\mathcal{D}^{s_2}\mathcal{Z})\dots(\mathcal{D}^{s_{L-1}}\mathcal{Z})(\mathcal{D}^{s_J}\mathcal{Z})\right),$$

where $s_1 + s_2 + \dots + s_{L-1} + s_L = s$. Leading twist is $J = 2$.

Twist Operators in $\mathcal{N} = 4$ and Spin Chains, II

At one loop, this is an integrable XXX Heisenberg chain with spin $= -\frac{1}{2}$. [Beisert, MS '03]. The Hamiltonian gives infinitely many rules for shifting spins from each site ℓ to the adjacent sites $\ell \pm 1$.

$$H = \sum_{\ell=1}^{L} \mathcal{H}_{\ell,\ell+1} \,.$$

$$\begin{aligned} \mathcal{H}_{1,2} \cdot (\mathcal{D}^{s_1} \mathcal{Z})\,(\mathcal{D}^{s_2} \mathcal{Z}) \quad &= \quad \Big(\psi(s_1+1) + \Psi(s_2+1) - 2\psi(1)\Big)(\mathcal{D}^{s_1} \mathcal{Z})\,(\mathcal{D}^{s_2} \mathcal{Z}) \\ &\quad - \sum_{\{s'\}} \frac{1}{|s'|}\,(\mathcal{D}^{s_1-s'} \mathcal{Z})\,(\mathcal{D}^{s_2+s'} \mathcal{Z}) \end{aligned}$$

The anomalous dimension Δ of these operators is related to the "energy" $E(s,g) = E_0(s) + g^2\,E_2(s) + g^4\,E_4(s) + \ldots$ of the spin chain through

$$\Delta(s,g) = s + L + g^2\,E(s,g) \quad \text{with} \quad g^2 \;=\; \frac{g_{\rm YM}^2\,N}{8\,\pi^2}\,.$$

For complex values of spin s related to the the pomeron (cf.Polchinski's talk). [Lipatov et.al.; Brower, Polchinski, Strassler, Tan, '06].

Magnons are Particles

$\mathcal{Z}$-particle = hole $\qquad$ $\mathcal{D}$-particle = magnon

$$|\mathcal{Z}\mathcal{Z}\mathcal{D}\mathcal{Z}\mathcal{Z}\mathcal{D}\mathcal{Z}\mathcal{Z}\mathcal{Z}\rangle = |..\mathcal{D}..\mathcal{D}...\rangle\,.$$

One-magnon States:

$$|\Psi\rangle = \sum_{1\leq\ell\leq L} \Psi(\ell)\ |...\mathcal{Z}\overset{\overset{\ell}{\downarrow}}{\mathcal{D}}\mathcal{Z}...\rangle\,,$$

Schrödinger equation $H\cdot|\Psi\rangle = E\,|\Psi\rangle$:

$$E\,\Psi(\ell) = 2\,\Psi(\ell) - \Psi(\ell-1) - \Psi(\ell+1)\,.$$

Fourier transforming, magnons start to "move",

$$\Psi(\ell) = e^{i\,p\,\ell}\,,$$

with the dispersion law $E = 2 - e^{i\,p} - e^{-i\,p}$, i.e.

$$\boxed{E = 4\sin^2\tfrac{p}{2}\,.}$$

This solves the one-magnon problem. What about many?

Magnon Scattering: The Bethe Ansatz

Two-magnon states:

$$|\Psi\rangle = \sum_{1\le\ell_1\le\ell_2\le L} \Psi(\ell_1,\ell_2)\ |\ldots\mathcal{Z}(\overset{\ell_1}{\downarrow}\!\!\mathcal{D}\mathcal{Z})\ldots\mathcal{Z}(\overset{\ell_2}{\downarrow}\!\!\mathcal{D}\mathcal{Z})\ldots\rangle\,.$$

First guess nearly works (even at $\ell_2 = \ell_1 + 1$):

$$\Psi(\ell_1,\ell_2) = e^{i\,p_1\,\ell_1 + i\,p_2\,\ell_2}\,, \qquad \boxed{E = \sum_{k=1}^{2} 4\sin^2\frac{p_k}{2}\,,}$$

except when the magnons collide at $\ell_2 = \ell_1$, where:

$$\begin{aligned} E\,\Psi(\ell_1,\ell_2) \ &= \ \tfrac{3}{2}\,\Psi(\ell_1,\ell_2) - \Psi(\ell_1-1,\ell_2) - \tfrac{1}{2}\,\Psi(\ell_1-1,\ell_2-1) + \\ &\quad + \tfrac{3}{2}\,\Psi(\ell_1,x_2) - \Psi(\ell_1,\ell_2+1) - \tfrac{1}{2}\,\Psi(\ell_1+1,\ell_2+1)\,. \end{aligned}$$

Problem fixed by Bethe's ansatz: [Bethe '31]

$$\boxed{\Psi(\ell_1,\ell_2) = e^{i\,p_1\,\ell_1 + i\,p_2\,\ell_2} + S(p_2,p_1)\,e^{i\,p_2\,\ell_1 + i\,p_1\,\ell_2}\,.}$$

Elementary algebra gives:

$$S(p_1,p_2) = -\frac{e^{ip_1+ip_2} - 2e^{ip_2} + 1}{e^{ip_1+ip_2} - 2e^{ip_1} + 1}\,.$$

We can think of $S(p_1,p_2)$ as an S-matrix. [Yang '67]

Bethe's Equations and Factorized Scattering

Periodic boundary conditions $\Psi(\ell_1, \ell_2) = \Psi(\ell_2, \ell_1 + L)$ yield the spectrum :

$$e^{ip_1 L} = S(p_1, p_2) \quad \text{and} \quad e^{ip_2 L} = S(p_2, p_1)\,.$$

The magic properties of this integrable Heisenberg magnet are verified when we try the Bethe ansatz for $s > 2$ magnons.

It still works! The multi-magnon terms in the Hamiltonian are crucial.

Periodic boundary conditions yield Bethe's equations:

$$e^{ip_k L} = \prod_{\substack{j=1 \\ j \neq k}}^{s} S(p_k, p_j)\,, \qquad E = \sum_{k=1}^{s} 4 \sin^2 \frac{p_k}{2}\,.$$

The S-matrix is factorized, and the spin chain is integrable.

Algebraic Bethe Ansatz

The reason for the exact solvability is, similar to the hydrogen atom, the existence of hidden conserved charges.

$$\boxed{[Q_i, Q_j] = 0\,.}$$

Among these are

$Q_1 = \exp iP =$ translation, $P = \sum p_k =$ momentum,

$Q_2 = E =$ energy.

These may be used to understand why the Bethe ansatz works, and the Bethe equations may be systematically deduced in an algebraic form:

$$\boxed{\left(\frac{u_k + \frac{i}{2}}{u_k - \frac{i}{2}}\right)^L = \prod_{\substack{j=1\\ j\neq k}}^{s} \frac{u_k - u_j - i}{u_k - u_j + i}\,, \qquad E = \sum_{k=1}^{s} \frac{1}{u_k^2 + \frac{1}{4}}\,.}$$

They are identical to the trigonometric version we had before after changing variables $u_k = \frac{1}{2}\cot\frac{p_k}{2}\,.$

A Message to our Condensed Matter Friends

All states are important.

That is, irrespectively of whether they are ferromagnetic or antiferromagnetic, of whether the length is large or small !

The Full Set of $8+8$ Magnons

The most natural vacuum of the full $\mathfrak{psu}(2,2|4)$ spin chain is the "ferromagnetic" BPS state

$$\mathrm{Tr}\,\mathcal{Z}^L$$

Apart from the magnon $\mathcal{D}$ we just discussed, there are many more, in fact $8+8$, corresponding to the transverse fluctuations of strings in the light-cone gauge. These should be: Four complex adjoint scalars $\mathcal{X}, \bar{\mathcal{X}}, \mathcal{Y}, \bar{\mathcal{Y}}$ and four light-cone covariant derivatives $\mathcal{D}, \bar{\mathcal{D}}, \mathcal{D}', \bar{\mathcal{D}}'$ as well as 8 adjoint fermions $\Psi_1, \ldots \Psi_8$.

Doping the BPS vacuum with the other kinds of magnons leads to further simple subsectors:

- $\mathfrak{su}(2)$ sector: $\mathrm{Tr}\,\mathcal{X}^M \mathcal{Z}^L + \ldots$

  This is the usual spin $+\frac{1}{2}$ Heisenberg magnet.

- $\mathfrak{su}(1|1)$ sector: $\mathrm{Tr}\,\Psi^M \mathcal{Z}^{L-M} + \ldots$

  This is a deformed XY model.

The S-matrix

It turns out that the S-matrix idea works beyond one loop, and the other types of magnons. It is a powerful tool to compare and relate the respective spectra of gauge theory and string theory via the AdS/CFT corrrespondence. [MS '04]

It allows, assuming integrability, to obtain spectral information in the absence of detailed knowledge of the underlying exact dilatation operator/Hamiltonian.

And indeed, the full S-matrix for all 16 magnons may be found, up to an unknown global phase, by symmetry considerations (cf. next talk by Niklas Beisert).

String Predictions for $\mathcal{N} = 4$

A key proposal for AdS/CFT: [Maldacena '97; Gubser,Klebanov, Polyakov; Witten '98]

$$\begin{array}{ccc} \text{string energy} & \leftrightarrow & \text{conformal dimension} \\ E & = & \Delta \end{array}$$

Simplifications occur in certain limits involving states with large angular momentum J on the five-sphere S^5, or large spin s on AdS. In the case of twist operators we may study both:

$$\mathrm{Tr}\,\mathcal{D}^s \mathcal{Z}^J$$

• BMN limit or *"plane wave limit"* [Berenstein,Maldacena,Nastase '02]
$J \gg 1, \quad s = 2, 3, \ldots$ Here, an effective, analytic expansion parameter seems to emerge:

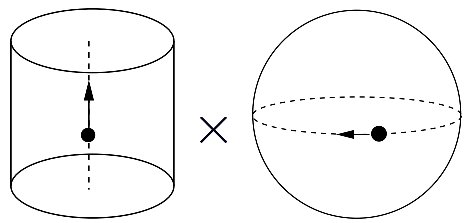
$$\lambda' = \frac{\lambda}{J^2}$$

In a moment we'll also look at the opposite limit:

• GKP limit [Gubser, Klebanov, Polyakov '02]
$s \gg 1, \quad J = 2, 3, \ldots$

The BMN Limit

In the BMN limit [Berenstein, Maldacena, Nastase '02] one simplifies the AdS/CFT correspondence and focuses on the geometry seen by a "tiny" string moving along a great circle on S^5:



It effectively sees a plane wave background with metric

$$ds^2 = -4\,dx^+dx^- - \mu^2\,(x^i)^2(dx^+)^2 + (dx^i)^2$$

The worldsheet theory becomes free, albeit massive: [Metsaev '01]

$$S_\mathrm{b} = \int d\tau d\sigma (\tfrac{1}{2}\,\partial_a x^i\,\partial^a x^i - \tfrac{\mu^2}{2}\,x^i\,x^i)$$

So quantization may be performed in a "textbook fashion", the spectrum is found explicitly, and yields predictions for conformal dimensions in $\mathcal{N}=4$ gauge theory.

BMN Scaling

This leads to the following prediction in the twist operator sector of the gauge theory, consisting of the complex scalars $\mathcal{Z}$ and the derivative $\mathcal{D}$:

$$\mathrm{Tr}\,\mathcal{D}^s \mathcal{Z}^J + \ldots \qquad \Rightarrow \qquad \Delta = J + \sum_{k=1}^{s} \sqrt{1 + \lambda' \, n_k^2}\,.$$

n_k are a mode numbers counting the various mixed states. Recall $\lambda' = \lambda / J^2$ with $J \gg 1$.

This is an all-loop prediction! It requires "BMN scaling". This means that at a given loop order ℓ the sum over all $\mathcal{O}(\lambda^\ell)$ Feynman diagrams has to scale at large R-charge J as $1/J^{2\ell}$!.

This is not true for individual Feynman diagrams.

Thus, a highly non-trivial property of gauge theory. Indirectly proven to 3-loop order. Is it true?

There are known counter examples such as the plane-wave matrix model where BMN scaling breaks down at four loops [Fischbacher,Klose,Plefka '04]. However, that model is not fully integrable [Beisert, Klose '05].

Deforming One-Loop Bethe Equations

I just told you about the one-loop Bethe equations for the twist operator sector:

$$\left(\frac{u_k + \frac{i}{2}}{u_k - \frac{i}{2}}\right)^J = \prod_{\substack{j=1 \\ j\neq k}}^{s} \frac{u_k - u_j - i}{u_k - u_j + i}\,, \qquad E_0 = \sum_{k=1}^{s} \frac{1}{u_k^2 + \frac{1}{4}}\,.$$

Recall also that this Bethe ansatz should yield the one-loop anomalous dimensions $\Delta(g) = J + s + g^2\,E_0$ of operators of the form

$$\mathrm{Tr}\,\mathcal{D}^s\mathcal{Z}^J + \ldots\,.$$

The all-loop "deformation" involves the (string-inspired [Kazakov, Marshakov, Minahan, Zarembo '04]) map [Beisert, Dippel, MS '04].

$$u = x + \frac{g^2}{2x}\,.$$

Higher-Loop Bethe Ansätze

Applying this Zhukovsky map

$$u = x + \frac{g^2}{2x}\,,$$

we proposed, with $x^\pm := x(u \pm \frac{i}{2})$, [Beisert, MS '05]

$$\left(\frac{x_k^+}{x_k^-}\right)^J = \prod_{\substack{j=1\\ j\neq k}}^{s} \frac{x_k^- - x_j^+}{x_k^+ - x_j^-}\,\frac{1 - g^2/2x_k^+ x_j^-}{1 - g^2/2x_k^- x_j^+}\,\sigma^2(u_k, u_j)\,,$$

with

$$e^{i\,p_k} = \frac{x_k^+}{x_k^-}\,, \qquad\qquad E = \sum_{k=1}^{s}\left(\frac{i}{x_k^+} - \frac{i}{x_k^-}\right).$$

Restriction: A priori only valid up to $\mathcal{O}(g^{2J-4})$! "Wrapping problem".

$\sigma^2(u_k, u_j)$ is the dressing factor. It is known to be one in gauge theory up to at least three loops: $\sigma^2 = 1 + \mathcal{O}(g^6)$. It is also known not to be one in string theory! Many workers in the field suspect/hope that it will differ from one at four loops ...

The Dispersion Law

This ansatz leads, via $e^{i\,p_k} = x_k^+/x_k^-$, to the dispersion law

$$E(g) = -\frac{M}{g^2} + \frac{1}{g^2}\sum_{k=1}^{M}\sqrt{1 + 8g^2\sin^2\frac{p_k}{2}}\,.$$

- It follows from basic structural properties of the long-range spin chain picture of perturbative gauge theory, in conjunction with the BMN result. [Beisert, Dippel, MS '04].
- It maybe derived, up to an unknown function $g^2 \leftrightarrow f(g^2)$ by combining the spin chain picture with supersymmetric representation theory. [Beisert '06].
- It also may be derived as an effective dispersion law from the Hubbard model. There the square roots are expressing the fact that this is an "effective" Bethe ansatz, not a fundamental one. [Rej, Serban, MS '05]

Transcendentality and Bethe Ansatz

In the case of twist-two operators wrapping is not a problem up to three loops. One can therefore find their anomalous dimensions from the Bethe ansatz. The result is reproduced by the expressions

$$
\begin{aligned}
E_0(s) &= 4\, S_1\,, \\
E_2(s) &= -4\left(S_3 + S_{-3} - 2\, S_{-2,1} + 2\, S_1\left(S_2 + S_{-2}\right)\right), \\
E_4(s) &= -8\Big(2\, S_{-3}\, S_2 - S_5 - 2\, S_{-2}\, S_3 - 3\, S_{-5} + 24\, S_{-2,1,1,1} \\
&\quad +6\left(S_{-4,1} + S_{-3,2} + S_{-2,3}\right) - 12\left(S_{-3,1,1} + S_{-2,1,2} + S_{-2,2,1}\right) \\
&\quad -\left(S_2 + 2\, S_1^2\right)\left(3\, S_{-3} + S_3 - 2\, S_{-2,1}\right) - S_1\left(8\, S_{-4} + S_{-2}^2\right. \\
&\quad \left. +4\, S_2\, S_{-2} + 2\, S_2^2 + 3\, S_4 - 12\, S_{-3,1} - 10\, S_{-2,2} + 16\, S_{-2,1,1}\right)\Big)\,,
\end{aligned}
$$

initially guessed by [Kotikov,Lipatov,Onishchenko,Velizhanin '04], and expressed in terms of recursively defined harmonic sums $(a, b, c > 0)$

$$
\begin{aligned}
S_{\pm a}(s) &= \sum_{m=1}^{s} \frac{(\pm 1)^m}{m^a}, \\
S_{\pm a,b,c,\cdots}(s) &= \sum_{m=1}^{s} \frac{(\pm 1)^m}{m^a}\, S_{b,c,\cdots}(m)\,.
\end{aligned}
$$

Kotikov-Lipatov Transcendentality and $\mathcal{N} = 4$

Kotikov and Lipatov obtained the corresponding two-loop result for $\mathcal{N} = 4$ gauge theory [Kotikov, Lipatov '03]. They noticed that the answer may be extracted from the QCD result by focusing on the "most complicated terms". These are the ones of highest degree of transcendentality.

Based on this experience with "translating" scaling dimensions at one and two loops from QCD to $\mathcal{N} = 4$, the above conjecture for the three-loop dimensions of the analog of these operators in (planar) $\mathcal{N} = 4$ was put forward by KLOV [Kotikov,Lipatov,Onishchenko,Velizhanin '04].

Two years ago a many-year effort to compute three-loop (NNLO) anomalous dimensions Δ of leading twist-two operators at finite spin s in QCD was completed.
[Moch,Vermaseren,Vogt '04]

The result fills pages ...

... and the "most complicated terms," i.e. the ones of highest degree of transcendentality, agree with the Bethe ansatz.

1 Results in Mellin space

Here we present the anomalous dimensions $\gamma_{\rm ns}^{\pm,\rm s}(N)$ in the $\overline{\rm MS}$-scheme up to the third order in the running coupling constant $\alpha_{\rm s}$, expanded in powers of $\alpha_{\rm s}/(4\pi)$. These quantities can be expressed in terms of harmonic sums [6,7,59,60]. Following the notation of [59], these sums are recursively defined by

$$S_{\pm m}(M) \;=\; \sum_{i=1}^{M} \frac{(\pm 1)^m}{i^m} \tag{1.1}$$

and

$$S_{\pm m_1,m_2,\ldots,m_k}(M) \;=\; \sum_{i=1}^{M} \frac{(\pm 1)^{m_1}}{i^{m_1}}\, S_{m_2,\ldots,m_k}(i)\;. \tag{1.2}$$

The sum of the absolute values of the indices m_k defines the weight of the harmonic sum. In the n-loop anomalous dimensions written down below one encounters sums up to weight $2n-1$.

In order to arrive at a reasonably compact representation of our results, we employ the abbreviation $S_{\vec{m}} \equiv S_{\vec{m}}(N)$ in what follows, together with the notation

$$\mathbf{N}_{\pm}\, S_{\vec{m}} \;=\; S_{\vec{m}}(N\pm 1)\;, \qquad \mathbf{N}_{\pm \mathbf{i}}\, S_{\vec{m}} \;=\; S_{\vec{m}}(N\pm i) \tag{1.3}$$

for arguments shifted by ± 1 or a larger integer i. In this notation the well-known one-loop (LO) anomalous dimension [1,2] reads

$$\gamma_{\rm ns}^{(0)}(N) \;\;=\;\; C_F\,(2(\mathbf{N}_- + \mathbf{N}_+)S_1 - 3)\;, \tag{1.4}$$

and the corresponding two second-order (NLO) non-singlet quantities [4,6] are given by

$$\begin{aligned}
\gamma_{\rm ns}^{(1)+}(N) \;&=\; 4C_AC_F\Big(2\,\mathbf{N}_+S_3 - \frac{17}{24} - 2S_{-3} - \frac{28}{3}S_1 + (\mathbf{N}_- + \mathbf{N}_+)\Big[\frac{151}{18}S_1 + 2S_{1,-2} - \frac{11}{6}S_2\Big]\Big) \\
&+ 4C_Fn_f\Big(\frac{1}{12} + \frac{4}{3}S_1 - (\mathbf{N}_- + \mathbf{N}_+)\Big[\frac{11}{9}S_1 - \frac{1}{3}S_2\Big]\Big) + 4{C_F}^2\Big(4S_{-3} + 2S_1 + 2S_2 - \frac{3}{8} \\
&+ \mathbf{N}_-\Big[S_2 + 2S_3\Big] - (\mathbf{N}_- + \mathbf{N}_+)\Big[S_1 + 4S_{1,-2} + 2S_{1,2} + 2S_{2,1} + S_3\Big]\Big)\;,
\end{aligned} \tag{1.5}$$

$$\gamma_{\rm ns}^{(1)-}(N) \;=\; \gamma_{\rm ns}^{(1)+}(N) + 16C_F\Big(C_F - \frac{C_A}{2}\Big)\Big((\mathbf{N}_- - \mathbf{N}_+)\Big[S_2 - S_3\Big] - 2(\mathbf{N}_- + \mathbf{N}_+ - 2)S_1\Big)\;. \tag{1.6}$$

The three-loop (NNLO, N^{2}LO) contribution to the anomalous dimension $\gamma_{\rm ns}^{+}(N)$ corresponding to the upper sign in Eq. (2.3) reads

$$\begin{aligned}
\gamma_{\rm ns}^{(2)+}(N) \;&=\; 16C_AC_Fn_f\Big(\frac{3}{2}\zeta_3 - \frac{5}{4} + \frac{10}{9}S_{-3} - \frac{10}{9}S_3 + \frac{4}{3}S_{1,-2} - \frac{2}{3}S_{-4} + 2S_{1,1} - \frac{25}{9}S_2 \\
&+ \frac{257}{27}S_1 - \frac{2}{3}S_{-3,1} - \mathbf{N}_+\Big[S_{2,1} - \frac{2}{3}S_{3,1} - \frac{2}{3}S_4\Big] - (\mathbf{N}_+ - 1)\Big[\frac{23}{18}S_3 - S_2\Big] - (\mathbf{N}_- + \mathbf{N}_+)\Big[S_{1,1} \\
&+ \frac{1237}{216}S_1 + \frac{11}{18}S_3 - \frac{317}{108}S_2 + \frac{16}{9}S_{1,-2} - \frac{2}{3}S_{1,-2,1} - \frac{1}{3}S_{1,-3} - \frac{1}{2}S_{1,3} - \frac{1}{2}S_{2,1} - \frac{1}{3}S_{2,-2} + S_1\zeta_3\;.
\end{aligned}$$

$$+\frac{1}{2}S_{3,1}\Big]\Big)+16C_F{C_A}^2\Big(\frac{1657}{576}-\frac{15}{4}\zeta_3+2S_{-5}+\frac{31}{6}S_{-4}-4S_{-4,1}-\frac{67}{9}S_{-3}+2S_{-3,-2}$$
$$+\frac{11}{3}S_{-3,1}+\frac{3}{2}S_{-2}-6S_{-2}\zeta_3-2S_{-2,-3}+3S_{-2,-2}-4S_{-2,-2,1}+8S_{-2,1,-2}-\frac{1883}{54}S_1$$
$$-10S_{1,-3}-\frac{16}{3}S_{1,-2}+12S_{1,-2,1}+4S_{1,3}-4S_{2,-2}-\frac{5}{2}S_4+\frac{1}{2}S_5+\frac{176}{9}S_2+\frac{13}{3}S_3$$
$$+(\mathbf{N}_-+\mathbf{N}_+-2)\Big[3S_1\zeta_3+11S_{1,1}-4S_{1,1,-2}\Big]+(\mathbf{N}_-+\mathbf{N}_+)\Big[\frac{9737}{432}S_1-3S_{1,-4}+\frac{19}{6}S_{1,-3}$$
$$+8S_{1,-3,1}+\frac{91}{9}S_{1,-2}-6S_{1,-2,-2}-\frac{29}{3}S_{1,-2,1}+8S_{1,1,-3}-16S_{1,1,-2,1}-4S_{1,1,3}-\frac{19}{4}S_{1,3}$$
$$+4S_{1,3,1}+3S_{1,4}+8S_{2,-2,1}+2S_{2,3}-S_{3,-2}+\frac{11}{12}S_{3,1}-S_{4,1}-4S_{2,-3}+\frac{1}{6}S_{2,-2}-\frac{1967}{216}S_2$$
$$+\frac{121}{72}S_3\Big]-(\mathbf{N}_--\mathbf{N}_+)\Big[3S_2\zeta_3+7S_{2,1}-3S_{2,1,-2}+2S_{2,-2,1}-\frac{1}{4}S_{2,3}-\frac{3}{2}S_{3,-2}-\frac{29}{6}S_{3,1}$$
$$+\frac{11}{4}S_{4,1}+\frac{1}{2}S_{2,-3}-S_{2,-2}\Big]+\mathbf{N}_+\Big[\frac{28}{9}S_3-\frac{2376}{216}S_2-\frac{8}{3}S_4-\frac{5}{2}S_5\Big]\Big)+16C_Fn_f^2\Big(\frac{17}{144}$$
$$-\frac{13}{27}S_1+\frac{2}{9}S_2+(\mathbf{N}_-+\mathbf{N}_+)\Big[\frac{2}{9}S_1-\frac{11}{54}S_2+\frac{1}{18}S_3\Big]\Big)+16{C_F}^2C_A\Big(\frac{45}{4}\zeta_3-\frac{151}{64}-10S_{-5}$$
$$-\frac{89}{6}S_{-4}+20S_{-4,1}+\frac{134}{9}S_{-3}-2S_{-3,-2}-\frac{31}{3}S_{-3,1}+2S_{-3,2}-\frac{9}{2}S_{-2}+18S_{-2}\zeta_3+10S_{-2,-3}$$
$$-6S_{-2,-2}+8S_{-2,-2,1}-28S_{-2,1,-2}+46S_{1,-3}+\frac{26}{3}S_{1,-2}-48S_{1,-2,1}+\frac{28}{3}S_{1,2}-\frac{185}{6}S_3$$
$$-8S_{1,3}+2S_{3,-2}-4S_5-(\mathbf{N}_-+\mathbf{N}_+-2)\Big[9S_1\zeta_3-\frac{133}{36}S_1+\frac{209}{6}S_{1,1}-14S_{1,1,-2}-\frac{242}{18}S_2$$
$$+9S_{2,-2}+\frac{33}{4}S_4-3S_{3,1}+\frac{14}{3}S_{2,1}\Big]+(\mathbf{N}_-+\mathbf{N}_+)\Big[17S_{1,-4}-\frac{107}{6}S_{1,-3}-32S_{1,-3,1}$$
$$-\frac{173}{9}S_{1,-2}+16S_{1,-2,-2}+\frac{103}{3}S_{1,-2,1}-2S_{1,-2,2}-36S_{1,1,-3}+56S_{1,1,-2,1}+8S_{1,1,3}$$
$$-\frac{109}{9}S_{1,2}-4S_{1,2,-2}+\frac{43}{3}S_{1,3}-8S_{1,3,1}-11S_{1,4}+\frac{11}{3}S_{2,2}+21S_{2,-3}-30S_{2,-2,1}-4S_{2,1,-2}$$
$$-5S_{2,3}-S_{4,1}+\frac{31}{6}S_{2,-2}-\frac{67}{9}S_{2,1}\Big]+(\mathbf{N}_--\mathbf{N}_+)\Big[9S_2\zeta_3+2S_{2,-3}+4S_{2,-2,1}-12S_{2,1,-2}$$
$$-2S_{2,3}+13S_{4,1}+\frac{1}{2}S_{2,-2}+\frac{11}{2}S_4-\frac{33}{2}S_{3,1}+\frac{59}{9}S_3+\frac{127}{6}S_{2,1}-\frac{1153}{72}S_2\Big]+\mathbf{N}_+\Big[8S_{3,-2}$$
$$+\frac{4}{3}S_{3,1}-2S_{3,2}+14S_5+\frac{23}{6}S_4+\frac{73}{3}S_3+\frac{151}{24}S_2\Big]\Big)+16{C_F}^2n_f\Big(\frac{23}{16}-\frac{3}{2}\zeta_3+\frac{4}{3}S_{-3,1}-\frac{59}{36}S_2$$
$$+\frac{4}{3}S_{-4}-\frac{20}{9}S_{-3}+\frac{20}{9}S_1-\frac{8}{3}S_{1,-2}-\frac{8}{3}S_{1,1}-\frac{4}{3}S_{1,2}+\mathbf{N}_+\Big[\frac{25}{9}S_3-\frac{4}{3}S_{3,1}-\frac{1}{3}S_4\Big]$$
$$-(\mathbf{N}_+-1)\Big[\frac{67}{36}S_2-\frac{4}{3}S_{2,1}+\frac{4}{3}S_3\Big]+(\mathbf{N}_-+\mathbf{N}_+)\Big[S_1\zeta_3-\frac{325}{144}S_1-\frac{2}{3}S_{1,-3}+\frac{32}{9}S_{1,-2}$$
$$-\frac{4}{3}S_{1,-2,1}+\frac{4}{3}S_{1,1}+\frac{16}{9}S_{1,2}-\frac{4}{3}S_{1,3}+\frac{11}{18}S_2-\frac{2}{3}S_{2,-2}+\frac{10}{9}S_{2,1}+\frac{1}{2}S_4-\frac{2}{3}S_{2,2}-\frac{8}{9}S_3\Big]\Big)$$
$$+16{C_F}^3\Big(12S_{-5}-\frac{29}{32}-\frac{15}{2}\zeta_3+9S_{-4}-24S_{-4,1}-4S_{-3,-2}+6S_{-3,1}-4S_{-3,2}+3S_{-2}+25S_3$$
$$-12S_{-2}\zeta_3-12S_{-2,-3}+24S_{-2,1,-2}-52S_{1,-3}+4S_{1,-2}+48S_{1,-2,1}-4S_{3,-2}+\frac{67}{2}S_2-17S_4$$

$$+(\mathbf{N}_- + \mathbf{N}_+ - 2)\Big[6S_1\zeta_3 - \frac{31}{8}S_1 + 35S_{1,1} - 12S_{1,1,-2} + S_{1,2} + 10S_{2,-2} + S_{2,1} + 2S_{2,2} - 2S_{3,1}$$
$$-3S_5\Big] + (\mathbf{N}_- + \mathbf{N}_+)\Big[23S_{1,-3} - 22S_{1,-4} + 32S_{1,-3,1} - 2S_{1,-2} - 8S_{1,-2,-2} - 30S_{1,-2,1} - 6S_{1,3}$$
$$+4S_{1,-2,2} + 40S_{1,1,-3} - 48S_{1,1,-2,1} + 8S_{1,2,-2} + 4S_{1,2,2} + 8S_{1,3,1} + 4S_{1,4} + 28S_{2,-2,1} + 4S_{2,1,2}$$
$$+4S_{2,2,1} + 4S_{3,1,1} - 4S_{3,2} + 8S_{2,1,-2} - 26S_{2,-3} - 2S_{2,3} - 4S_{3,-2} - 3S_{2,-2} - 3S_{2,2} + \frac{3}{2}S_4\Big]$$
$$+(\mathbf{N}_- - \mathbf{N}_+)\Big[12S_{2,1,-2} - 6S_2\zeta_3 - 2S_{2,-3} + 3S_{2,3} + 2S_{3,-2} - \frac{81}{4}S_{2,1} + 14S_{3,1} - 5S_{2,-2}$$
$$-\frac{1}{2}S_{2,2} + \frac{15}{8}S_2 + \frac{1}{2}S_3 - 13S_{4,1} + 4S_5\Big] + \mathbf{N}_+\Big[14S_4 - \frac{265}{8}S_2 - \frac{87}{4}S_3 - 4S_{4,1} - 4S_5\Big]\Big) \; .$$

The third-order result for the anomalous dimension $\gamma_{\rm ns}^{-}(N)$ corresponding to the lower sign in Eq. (2.3) is given by

$$\gamma_{\rm ns}^{(2)-}(N) \;=\; \gamma_{\rm ns}^{(2)+}(N) + 16C_AC_F\Big(C_F - \frac{C_A}{2}\Big)\Big((\mathbf{N}_- + \mathbf{N}_+ - 2)\Big[\frac{367}{18}S_1 + 12S_1\zeta_3 + 2S_{1,-2}$$
$$+4S_{1,-3} + 8S_{1,-2,1} + \frac{140}{3}S_{1,1} - 16S_{1,1,-2} - S_5 - 8S_{3,1} - S_4\Big] + (\mathbf{N}_- - \mathbf{N}_+)\Big[4S_5 - 12S_2\zeta_3$$
$$-4S_{2,-3} - 8S_{2,-2,1} - \frac{70}{3}S_{2,1} + 16S_{2,1,-2} + 4S_{3,-2} - 8S_{4,1} + \frac{70}{3}S_{3,1} + \frac{13}{3}S_4 - \frac{41}{18}S_2$$
$$+2S_{2,-2} - \frac{152}{9}S_3\Big] + 4(\mathbf{N}_+ - 1)\Big[4S_{2,-2} - 8S_2 - S_3\Big]\Big) + 16C_Fn_f\Big(C_F - \frac{C_A}{2}\Big)$$
$$\cdot\Big((\mathbf{N}_- + \mathbf{N}_+ - 2)\Big[\frac{61}{9}S_1 - \frac{8}{3}S_{1,1}\Big] + (\mathbf{N}_- - \mathbf{N}_+)\Big[\frac{4}{3}S_{2,1} - \frac{41}{9}S_2 + \frac{38}{9}S_3 - \frac{4}{3}S_{3,1} - \frac{4}{3}S_4\Big]\Big)$$
$$+16{C_F}^2\Big(C_F - \frac{C_A}{2}\Big)\Big((\mathbf{N}_- + \mathbf{N}_+ - 2)\Big[8S_{1,-2} - 15S_1 - 12S_1\zeta_3 - 12S_{1,-3} - 60S_{1,1}$$
$$+24S_{1,1,-2} + 8S_{1,2} + 40S_2 - 12S_{2,-2} + 8S_{2,1} + 7S_3 + 12S_{3,1} + 6S_5\Big] + (\mathbf{N}_- - \mathbf{N}_+)\Big[12S_2\zeta_3$$
$$-24S_2 + 12S_{2,-3} + 8S_{2,-2} + 30S_{2,1} - 24S_{2,1,-2} - 4S_{2,2} - 15S_3 - 38S_{3,1} + 4S_{3,2} + 24S_{4,1}$$
$$-12S_5\Big] - (\mathbf{N}_+ - 1)\Big[8S_{3,-2} + 26S_4\Big]\Big) \; .$$

Finally the quantity $\gamma_{\rm ns}^{\rm s}(N)$ corresponding to the last term in Eq. (2.5) starts at three loops with

$$\gamma_{\rm ns}^{(2)\rm s}(N) \;=\; 16n_f\frac{d^{abc}d_{abc}}{n_c}\Big((\mathbf{N}_- + \mathbf{N}_+)\Big[\frac{25}{3}S_1 + \frac{11}{12}S_{1,-3} - \frac{5}{3}S_{1,-2,1} - \frac{1}{6}S_{1,1,-2}\Big]$$
$$+(\mathbf{N}_- + \mathbf{N}_+ - 2)\Big[\frac{13}{12}S_{1,-2} + \frac{91}{24}S_{1,1} - \frac{3}{8}S_{1,3} - \frac{1}{4}S_{2,-2} - \frac{91}{48}S_2 + \frac{3}{16}S_3 + \frac{5}{8}S_{3,1}\Big]$$
$$+\frac{2}{3}(\mathbf{N}_+ - \mathbf{N}_{+2})\Big[S_4 + S_{2,-2} - S_{3,1}\Big] - \frac{2}{3}(\mathbf{N}_{-2} + \mathbf{N}_{+2})\Big[S_{1,-3} - S_{1,-2,1} - S_{1,1,-2}\Big]$$
$$+(\mathbf{N}_- - 1)\Big[\frac{1}{4}S_4 + \frac{1}{2}S_5\Big] + (\mathbf{N}_- - \mathbf{N}_+)\Big[\frac{1}{2}S_{2,-3} + \frac{1}{2}S_{2,-2} - \frac{109}{48}S_2 - \frac{41}{24}S_{2,1} + \frac{67}{48}S_3$$
$$-\frac{1}{2}S_{3,1} - S_{2,1,-2} + \frac{1}{4}S_{2,3} + \frac{1}{2}S_{3,-2} - \frac{3}{4}S_{4,1}\Big] - \frac{50}{3}S_1 - \frac{1}{2}S_{1,-3} + 2S_{1,-2,1} - S_{1,1,-2}\Big) \; .$$

Transcendentality, Solvability and Integrability

This structural beauty, and (relative) simplicity, when comparing $\mathcal{N} = 4$ to QCD, is one of many hints on a hidden solvable structure in planar $\mathcal{N} = 4$ gauge theory!

So we can say that at least a part of a real world QCD answer (a four-dimensional gauge theory!) may be explained by two-dimensional integrable structures!

Also, recall that this Bethe ansatz was found by comparing gauge and string theory via the AdS/CFT correspondence!

Good ... but not all is well with AdS/CFT just yet ...

The Dressing Factor, I

The dressing factor σ^2 [Arutyunov, Frolov, MS '04] was proposed as a way to repair the infamous discrepancies between certain semi-classical string states (near-BMN, Frolov-Tseytlin) and "long" gauge theory states [Callan,Lee,McLoughlin,Schwarz,Swanson,Wu '03], [Serban,MS '04]. We have, to leading semi-classical order,

$$\sigma(u_k, u_j) = \prod_{r=2}^{\infty} \exp\left[i \left(\frac{g^2}{2}\right)^r \left(q_r(u_k)\, q_{r+1}(u_j) - q_r(u_j)\, q_{r+1}(u_k)\right)\right]$$

Here the charges $q_r(u)$ are the eigenvalues of the hidden set of conserved charges Q_r of the deformed integrable model, which mutually commute: $[Q_r, Q_{r'}] = 0$.

Open Problem:

Where does this interesting analytical structure come from?

The Dressing Factor, II

This ansatz is only believed to be valid at leading semi-classical order, and we expect quantum corrections.

[Arutyunov, Frolov, MS '04, Beisert, Tseytlin '05, Schafer-Nameki, Zamaklar '05]

The next-order, one-loop string theory corrections were indeed recently derived [Hernández, López '06], and independently checked in [Freyhult, Kristjansen '06]. They read

$$\sigma(u_k, u_j) = \prod_{r=2}^{\infty} \prod_{n=0}^{\infty} e^{i\, c_{r,r+1+2n}\left(q_r(u_k)\, q_{r+1+2n}(u_j) - q_r(u_j)\, q_{r+1+2n}(u_k)\right)}$$

where

$$c_{r,s} = \left(\frac{\lambda}{16\pi^2}\right)^r \left(\delta_{r+1,s} - \frac{1}{\sqrt{\lambda}} \frac{8\,(r-1)(s-1)}{(r+s-2)(s-r)} + \mathcal{O}(\frac{1}{\lambda})\right)$$

How to find the exact expression?

Maybe using crossing symmetry? [Janik '06]
The quantum-corrected σ^2 satisfies this equation

[Arutyunov, Frolov '06].

And does the exact result correctly reproduce the gauge theory result $\sigma = 1 + \mathcal{O}(g^6)$?

Features if $\sigma^2 \neq 1$

- It allows to explain everything that is currently known about the string spectrum: Semiclassical strings (Frolov-Tseytlin, Gubser-Klebanov-Polyakov, $\lambda^{\frac{1}{4}}$-behavior, near-BMN limit, Hofman-Maldacena giant magnons, ...)

- It necessarily leads to a breakdown of perturbative BMN-scaling at weak coupling at four loops or beyond.

- It necessarily leads to a breakdown of Lipatov's transcendentality conjecture at four loops or beyond.

- It is only known approximately. Currently little indication, but not excluded, that it is able to reproduce reasonable weak coupling behavior.

- It is however indicative of a non-trivial vacuum structure of the BPS-states. Vacuum appears to be polarized with particle-antiparticle pairs. [Minahan '05, Beisert '06, Janik '06].

Features if $\sigma^2 = 1$

- It leads to all-order perturbative BMN-scaling at weak coupling.
- It reproduces Lipatov's transcendentality conjecture to all orders in perturbation theory.
- In the $\mathfrak{su}(2)$ sector a microscopically well-defined model (the Hubbard model) exists which yields the asymptotic Bethe equations with $\sigma^2 = 1$ to all orders in perturbation theory. [Rej, Serban, MS '05]
- It is indicative of a trivial "reference" vacuum structure of the BPS-states. No particle-antiparticle vacuum polarization.

The Mystery of the Dressing Factor

Are there hidden non-perturbative effects in $N = \infty$ gauge theory, despite convergence of planar perturbation theory?

Is there a hidden, dynamical structure in the vacuum states, i.e. the BPS states $\mathrm{Tr}\, Z^J$ of $\mathcal{N} = 4$ gauge theory?

Or, finally, is there a large N "Gross-Witten" phase transition in AdS/CFT as one goes from weak to strong coupling?

Transcendentality might elucidate this mystery ... let's see why.

The Large-Spin Scaling Function

The anomalous dimension of operators of low twist behaves in a very interesting, logarithmic way at large spin $s \to \infty$:

$$\Delta = s + f(g) \log(s) + O(s^0) .$$

Here $f(g)$ is the scaling function, sometimes called "cusp" anomalous dimension, as it also controls the divergences of cusps of light-like Wilson loops.

In QCD, $f(g)$ can be probed experimentally for small g in deep inelastic scattering.

This logarithmic behavior is quite a miracle, as individual Feynman diagrams contain divergences of the type $\log^k(s)$. The KLOV result reproduces this feature:

$$\boxed{f(g) = 4\, g^2 - \tfrac{2}{3}\, \pi^2\, g^4 + \tfrac{11}{45}\, \pi^4\, g^6 + \ldots}$$

The transcendentality principle also extends to large s, as can be seen when writing the result in the form

$$\boxed{f(g) = 4\, g^2 - 4\, \zeta(2)\, g^4 + \Big(4\, \zeta(2)^2 + 12\, \zeta(4)\Big)\, g^6 + \ldots}$$

Integrability and Transcendentality

Now we can use our all-loop Bethe ansatz for twist operators [Beisert, MS '05] and get a prediction for the scaling function beyond three loops. Let us first assume a dressing factor $\sigma^2 = 1$.

While asymptotic, they should be fine for studying the large spin s limit of twist-two operators. The reason is that the lowest state of finite twist operators is believed to scale independently of the twist J, as long as $J \ll s$.

The iteration of the non-singular Fredholm integral equation derived from the Bethe equations yields at small g [Eden,MS '06]

$$
\begin{aligned}
f(g) = {} & \\
& 4\,g^2 - 4\,\zeta(2)\,g^4 + \Big(4\,\zeta(2)^2 + 12\,\zeta(4)\Big)\,g^6 \\
& - \Big(4\,\zeta(2)^3 + 24\,\zeta(2)\zeta(4) - 4\,\zeta(3)^2 + 50\,\zeta(6)\Big)\,g^8 \\
& + \Big(4\,\zeta(2)^4 + 36\,\zeta(2)^2\zeta(4) - 8\,\zeta(2)\zeta(3)^2 \\
& \qquad + 100\,\zeta(2)\zeta(6) - 40\,\zeta(3)\zeta(5) + 39\,\zeta(4)^2 + 245\,\zeta(8)\Big)\,g^{10} \\
& + \ldots
\end{aligned}
$$

The Kotikov-Lipatov transcendentality principle is realized!
Also note that all coefficients are integers. [Lipatov,Kotikov '03].

A Novel Four-Loop Prediction

The result may be simplified, which however obscures the integer nature of the coefficients. To four-loop order we have

$$f(g) = 4\,g^2 - \tfrac{2}{3}\,\pi^2\,g^4 + \tfrac{11}{45}\,\pi^4\,g^6 - \left(\tfrac{73}{630}\,\pi^6 - 4\,\zeta(3)^2\right)\,g^8 + \ldots$$

This agrees to three loops with [Bern, Dixon, Smirnov '05], and with the earlier, finite spin s results of [Kotikov,Lipatov,Onishchenko,Velizhanin '04].

The four-loop prediction is new, and will hopefully be tested in the nearest future with the iterative gluon-amplitude approach [Bern, Czakon, Dixon, Kosower, Smirnov, work in progress], cf. D. Kosowers's talk.

However, if it breaks down, it will lead to a simultaneous breakdown of BMN-scaling and transcendentality. These are intricately linked through the structure of the Bethe ansatz. This "simultanenous" breakdown can be detected by the following modification of the four-loop term:

$$-\left(\frac{73}{630}\,\pi^6 - 4\,\zeta(3)^2 + 8\,\beta\,\zeta(3)\right)\,.$$

Here β is some number which is with some likelyhood rational.

Soft Breaking of BMN-Scaling and Maximal Transcendentality

The reason why such a statement is possible is the intricate way the Bethe ansatz of the model links all sectors through supersymmetry. The only way the current gauge theory equations may get modified is through the dressing factor. The only form that is currently consistent with what we know at weak coupling is [Beisert, Klose '05],

$$\sigma^2(u_k, u_j) = e^{i\,\beta\, g^6\, \left(q_2(u_k) q_3(u_j) - q_3(u_k) q_2(u_j)\right) + \ldots}\,.$$

leading to the above modification, while breaking BMN scaling. Incidentally this is of AFS string dressing factor form [Arutyunov, Frolov, MS '04], but with $g^4 \to g^6$.

Such a modified Bethe ansatz would also change the anomalous dimensions of all "short" operators. E.g. for $\mathrm{Tr}\, X^2\, Z^3 + \ldots$

$$E(g) = 4 - 6\, g^2 + 17\, g^4 - (\frac{115}{2} - 8\, \beta)\, g^6 \, + \ldots\,.$$

It would be a bit strange if the modification were non-rational, as there also seems to be a minimal transcendentality principle for short operators. But of course $\beta \sim \zeta(3)$ *could* appear ...

Outlook

- Can we find the full, non-asymptotic (i.e. including short operators), perturbative spectrum of $\mathcal{N} = 4$ gauge theory ?

- Can we, exploiting integrability, quantize the string σ-model, and derive an exact Bethe ansatz for strings on $AdS_5 \times S^5$? Exact means exact in λ, and non-asymptotic!

- Are there yet-to-be-discovered non-perturbative effects in planar $\mathcal{N} = 4$ gauge theory? If so, do they reconcile gauge and string theory?

- Or do perturbative BMN-scaling and maximal transcendentality break down at four loops, or beyond? Neat features, but maybe not "fundamental"? Is this what AdS/CFT is forcing us to conclude?

- Can we relate the "worldsheet" S-matrix to the "space-time" S-matrix (in gauge and string theory) ? Is the planar AdS/CFT system not only integrable (a precise but narrow notion which only relates to the spectrum of operator dimensions) but, more generally, solvable?

Final Remark

- It is hoped that having an exactly solvable example for a gauge/string duality will lead to deep insights into the fundamental nature of string theory and quantum gravity. Important applications of this activity to QCD and, surprisingly, condensed matter theory are likely.