On the spectrum of strings in $AdS_5 \times S^5$

J.A. Minahan, A. Tirziu and A.A.T.

“Infinite spin limit of semiclassical string states,” hep-th/0606145

M. Kruczenski, J. Russo and A.A.T.

“Spiky strings and giant magnons on S5,” hep-th/0607044

also

R. Roiban, A. Tirziu and A.A.T.

“Asymptotic Bethe Ansatz S-matrix and Landau-Lifshitz type effective 2-d actions,” hep-th/0604199

also talks by Zarembo, Klebanov, and more to follow

AdS/CFT

$\mathcal{N} = 4$ SYM at $N = \infty$

dual to type IIB superstrings in $AdS_5 \times S^5$

Parameters:

$\lambda = g^2_{_{YM}} N$ related to string tension

$2\pi T = \frac{R^2}{\alpha'} = \sqrt{\lambda}$

$g_s = \frac{\lambda}{4\pi N} \to 0$

string energies = dimensions of gauge-invariant operators

$$E(\sqrt{\lambda}, J, m, ...) = \Delta(\lambda, J, m, ...)$$

J - global charges of $SO(2,4) \times SO(6)$:

spins $S_1, S_2;\ J_1, J_2, J_3$

m - windings, folds, cusps, oscillation numbers, ...

Operators: $\text{Tr}\big(\Phi_1^{J_1} \Phi_2^{J_2} \Phi_3^{J_3} D_+^{S_1} D_\perp^{S_2} ... F_{mn} ... \Psi ...\big)$

Solve susy 4-d CFT = string in R-R background:

compute $E = \Delta$ for any λ (and J,m)

Perturbative expansions are opposite:

$\lambda \gg 1$ in perturbative string theory

$\lambda \ll 1$ in perturbative planar gauge theory

"Constructive" approach:

use perturbative results on both sides and other properties (integrability, susy,+?) to guess exact answer (Bethe ansatz,...)

Remarkable recent progress:

– "semiclassical" states with large quantum numbers

dual to "long" gauge operators

$E = \Delta$ – same dependence on $J, m, ...$

coefficients = interpolating functions of λ

– connection to spectrum of integrable spin chains

– advances in uncovering underlying Bethe ansatz

String Theory in $AdS_5 \times S^5$

$$S = T \int d^2\sigma \Big[G_{mn}(x) \partial x^m \partial x^n + \bar\theta (D + F_5) \theta \partial x$$
$$+ \bar\theta\theta\bar\theta\theta \partial x \partial x + ... \Big]$$

$T = \frac{R^2}{2\pi\alpha'} = \frac{\sqrt{\lambda}}{2\pi}$ (Metsaev, AT 98)

Conformal invariance: $\beta_{mn} = R_{mn} - (F_5)^2_{mn} = 0$

Classical integrability (Bena, Polchinski, Roiban 02)

Progress in detailed understanding of implications of (semi)classical integrability (Kazakov, Marshakov, Minahan, Zarembo 04; Beisert, Kazakov, Sakai, Zarembo 05; Dorey, Vacedo 06,...)

Explicit computation of 1-loop quantum superstring ($1/T$) corrections to classical string energies (Frolov, AT 02-4, ...)

Near-geodesic expansion (Parnachev, Ryzhov; Callan, Lee, McLoughlin, Schwarz, Swanson, Wu 03; ...)

1-loop S-matrix? Beyond 1-loop? Quantum integrability?

$\mathcal{N} = 4$ Conformal Gauge Theory

Dimensions of operators: eigenvalues of dilatation operator

e.g., operators built out of SYM scalars (dual to strings in S^5)

$SU(2)$ sector: Tr$(\Phi^{J_1}\Phi^{J_2}) + \ldots$, $J = J_1 + J_2$

$\Phi_1 = \phi_1 + i\phi_2, \quad \Phi_2 = \phi_3 + i\phi_4$

planar 1-loop dilatation operator of $\mathcal{N} = 4$ SYM:

= Hamiltonian of ferromagnetic Heisenberg XXX$_{1/2}$ spin chain (Minahan, Zarembo 02):

$$H_1 = \frac{\lambda}{(4\pi)^2} \sum_{l=1}^{J} (I - \vec{\sigma}_l \cdot \vec{\sigma}_{l+1})$$

Higher orders (Beisert, Kristjansen, Staudacher 03; Beisert 04; Eden, Jarczak, Sokatchev 04):

$H_2 = \frac{\lambda^2}{(4\pi)^4} \sum_{l=1}^{J} (-3 + 4\vec{\sigma}_l \cdot \vec{\sigma}_{l+1} - \vec{\sigma}_l \cdot \vec{\sigma}_{l+2})$

H_3 contains $\vec{\sigma}_l \cdot \vec{\sigma}_{l+3}$ but also $(\vec{\sigma}_l \cdot \vec{\sigma}_{l+1})(\vec{\sigma}_{l+2} \cdot \vec{\sigma}_{l+3})$, etc.

operator dimensions = eigenvalues of “long-range” ferromagnetic spin chain H with “multi-spin” interactions

H_{eff} for Hubbard model (at least to 3 loop order) (Rej, Serban, Staudacher 05)

Spectrum? Compare to string theory?

Integrability (!) $\longrightarrow$ Bethe ansatz $\longrightarrow$ Spectrum

1-loop: Heisenberg model $\longrightarrow$ Bethe ansatz equations:

$$e^{ip_k J} = \prod_{j\neq k}^{M} \frac{u_k - u_j + i}{u_k - u_j - i} \, ,$$

$u_j = \frac{1}{2} \cot \frac{p_j}{2} \, , \qquad J = J_1 + J_2 \, , \qquad M = J_2$

$E = J + \frac{\lambda}{\pi^2} \sum_{j=1}^{M} \sin^2 \frac{p_j}{2}, \qquad \sum_{j=1}^{M} p_j = 2\pi m$

Indications of integrability of both string ($\lambda \gg 1$) and gauge ($\lambda \ll 1$) theory: expect Bethe ansatz description for any λ

(Beisert, Dippel, Staudacher 04)

$$e^{ip_k J} = \prod_{j\neq k}^{M} S(p_k, p_j; \lambda), \qquad S = S_1 \, e^{i\theta}$$

$S_1 = \frac{u_k - u_j + i}{u_k - u_j - i} \, , \qquad \theta = \theta(p_k, p_j; \lambda)$

$u_j(p_j, \lambda) = \frac{1}{2} \cot \frac{p_j}{2} \sqrt{1 + \frac{\lambda}{\pi^2} \sin^2 \frac{p_j}{2}}$

p_j for bound states with $\sum_{k=1}^{M} p_k = 2\pi m$

$E = J + \sum_{j=1}^{M} \left(\sqrt{1 + \frac{\lambda}{\pi^2} \sin^2 \frac{p_j}{2}} - 1 \right)$

S = phase shift due to magnon scattering (Staudacher 05)

What about θ ?

Perturbative gauge theory: “Asymptotic” BDS ansatz

$J \to \infty$, up to λ^J order: $\quad S = S_1, \;\; \theta = 0$

But to match semiclassical string theory need $\theta \neq 0$

Perturbative string theory: “String” AFS ansatz

(Arutyunov, Frolov, Staudacher 04)

θ – common to all sectors, structure fixed by symmetries

(Beisert 05)

$$\theta(p, p'; \lambda) = \sum_{r=2}^{\infty} \sum_{s=r+1}^{\infty} c_{rs}(\lambda)[q_s(p')q_r(p) - q_s(p)q_r(p')]$$

$$q_{r+1}(p) = \tfrac{2}{r} \sin \tfrac{rp}{2} \Big(\frac{\sqrt{1+4\bar\lambda \sin^2 \frac{p}{2}} - 1}{\bar\lambda \sin \frac{p}{2}}\Big)^r , \qquad \bar\lambda \equiv \frac{\lambda}{(2\pi)^2}$$

Matching to classical string:

$(c_{rs})_{\lambda \to \infty} \;\to\; \lambda^{\frac{r+s-1}{2}} \delta_{r,s-1}$ (AFS)

at large λ expect from string theory

$$c_{rs}(\lambda \gg 1) = \bar\lambda^{\frac{r+s-1}{2}} [\delta_{r,s-1} + \tfrac{1}{\sqrt{\bar\lambda}} a_{rs} + \tfrac{1}{(\sqrt{\bar\lambda})^2} b_{rs} + ...]$$

$$c_{rs}(\lambda \ll 1) \;\to\; 0 \;\;\; ?$$

Compute $c_{rs}(\lambda)$ from "first principles"

– from quantum $AdS_5 \times S^5$ superstring

String 1-loop corrections to string energies (Frolov, AT 03; Park, Tirziu, AT 05) imply $a_{rs} \neq 0$ (Beisert, AT 05)

1-loop string results translate into (Hernandez, Lopez 06)

$$a_{rs} = \frac{2}{\pi}[1 - (-1)^{r+s}]\frac{(r-1)(s-1)}{(r-1)^2-(s-1)^2}$$

Consistent (Arutyunov, Frolov 06; Beisert 06) with crossing condition (Janik 06)

Beyond 1-loop order? Which are additional constraints?

Various Attempts:

• compute S-matrix directly from superstring theory

Important conceptual role played by non-relativistic "Landau-Lifshitz" type effective action for positive energy magnons (Kruczenski 03)

S-matrix of magnons with "non-relativistic" dispersion relation (Klose, Zarembo 06)

S= effective string theory S-matrix of "positive-energy" branch of BMN-type string modes: "integrate out" negative-energy branch (Roiban, Tirziu, AT 06)

• String sigma model (in conformal gauge): suggests interpret S as “effective” scattering matrix of integrable Lorentz-invariant 2d field theory whose effective excitations correspond to spin chain magnons (Polchinski, Mann 05; Gromov, Kazakov, Sakai, Viera 06; Gromov, Kazakov 06)

• detailed study of spectrum in various limits on gauge and string sides $\longrightarrow$ extra constraints on S-matrix

Key assumption:

Expect spectrum to have qualitatively same structure at any λ (at least for large J)

smooth change with λ: no transition on the way from small to large λ

Indeed, remarkable evidence (qualitative and quantitative) of correspondence between string and gauge states

sometimes works better than one could expect (susy: non-renormalization of some coefficients, ...)

Plan:

compare weak-coupling spin chain spectrum with semiclassical string spectrum

Gauge theory spectrum at $\lambda \ll 1$ and $J \gg 1$

1-loop: $\text{XXX}_{1/2}$ Heisenberg, length $J = J_1 + J_2$, solve BA

energy $E - J = \lambda E_1 [1 + O(\frac{1}{J})] + O(\lambda^2)$

- $E_1 = 0$: ferromagnetic vacuum (BPS operator Tr Φ^J)
- $E_1 = \frac{a}{J^2}$: $J_2 = 2$, magnons

$p = \frac{2\pi n}{J}$, $w \sim p^2$: BMN operators

$\sum e^{ipJ} \text{Tr}([\Phi_1 .... \Phi_1] \Phi_2 [\Phi_1 .... \Phi_1] \Phi_2 ...)$

- $E_1 = \frac{b}{J}$: $J_1 \sim J_2 \gg 1$, low-energy spin waves

"Thermodynamic" limit: bound states of

large number ($J_2 \sim J \gg 1$) of magnons, $b = b(\frac{J_2}{J})$

"Bloch walls" or "macroscopic Bethe strings" (Sutherland 95; Dhar, Shastry 00; Beisert, Minahan, Staudacher, Zarembo 03); "locally BPS" operators

Tr$([\Phi_1 .... \Phi_1][\Phi_2 ... \Phi_2][\Phi_1 .... \Phi_1][\Phi_2 ... \Phi_2] ...)$

- $E_1 = c$: bound states of finite no. of magnons

"Bethe strings" (Bethe 31), $c \sim \frac{1}{J_2}$

- $E_1 = kJ$: antiferromagnetic ($J_1 = J_2 \gg 1$) state

$k = \frac{\ln 2}{4\pi^2}$ (Huelthen 38) same structure of semiclassical spectrum on string side

Bethe bound states of magnons

Limit: $J_1 \gg J_2$, e.g., $J_1 \to \infty,\ J_2$=finite

$J \to \infty$ with complex $p_j = a_j + ib_j$: solutions related to poles (or zeroes) of the S-matrix

$$u_j = u_0 + i[\frac{1}{2}(J_2 + 1) - j], \qquad j = 1, ..., J_2,$$

$u_0 = \text{real}\ , \qquad u_j = \frac{1}{2}\cot\frac{p_j}{2}$

"Bethe string":

*

*

——u_0

*

*

$$E - J = \frac{\lambda}{2\pi^2}\sum_1^{J_2}\sin^2\frac{p_j}{2} \ = \ \frac{\lambda}{2\pi^2}\frac{1}{J_2}\sin^2\frac{p}{2}$$

$$p = \sum_1^{J_2} p_j = \ \pi - 2\arctan\frac{2u_0}{J_2}$$

When J_2 grows to become of order J strings bend: become "macroscopic strings": $E - J = \frac{\lambda}{J}\, b(\frac{J_2}{J})$

Effective field theory: Landau Lifshitz model

part of the spectrum approximated by low-energy 2d effective action: slow modes at large J

important "bridge" to string-theory picture (Kruczenski 03)

spin coherent states $U^\dagger \vec{\sigma} U = \vec{n}, \quad \vec{n}^2 = 1$

Discrete path integral action:

$$S = \int dt \sum_{l=1}^{J} \left[\vec{C}(n_l) \cdot \partial_t \vec{n}_l - \frac{\lambda}{2(4\pi)^2} (\vec{n}_{l+1} - \vec{n}_l)^2 \right]$$

$\vec{n} = (\sin\psi\ \cos\varphi,\ \sin\psi\ \sin\varphi,\ \cos\psi)$

$dC = \epsilon^{ijk} n_i dn_j \wedge dn_k \ , \qquad \vec{C} \cdot d\vec{n} = \cos\psi\, d\varphi$

large J limit and low-energy excitations: n_l change slowly – continuum limit – $\vec{n}(t,\sigma) = \{\vec{n}(t, \frac{2\pi}{J} l)\}, \ \ l = 1, ..., J$

$$S = J \int dt \int_0^{2\pi} d\sigma\ L\ , \quad L = \vec{C} \cdot \partial_t \vec{n} - \frac{1}{8} \tilde{\lambda} (\partial_\sigma \vec{n})^2\ ,$$

$L = \cos\psi\, \dot{\varphi} - \frac{1}{8} \tilde{\lambda} (\psi'^2 + \sin^2\psi\, \varphi'^2)\ , \qquad \tilde{\lambda} \equiv \frac{\lambda}{J^2}$

Landau-Lifshitz equations of motion $\dot{n}_i = \frac{1}{2} \tilde{\lambda} \epsilon_{ijk} n_j n''_k$:
Integrable system: Lax pair, inverse scattering method, etc.

LL model on a circle:

•magnons: small fluctuations near $\vec{n} = (0,0,1)$

$n_1 + in_2 \sim e^{iwt+in\sigma}, \qquad w \sim \bar{\lambda} p^2 \ , \ \ p = \frac{2\pi n}{J}$

•solitons: finite $E - J \sim \frac{\lambda}{J}, \ \ J_2 \sim J$

same as (semiclassical limit of) "macroscopic strings", e.g.

$\psi = m\sigma, \ \varphi =$const

what about $J \gg 1, \ J_2$=fixed states?

LL model on a line:

rescale $\sigma \to x = J\sigma$ and take $J \to \infty$,

$S = \int dt \int_{-\infty}^{\infty} dx \ L$

$L = \cos\psi \ \dot{\varphi} - \frac{1}{8}\bar{\lambda}(\psi'^2 + \sin^2\psi \ \varphi'^2) \ , \qquad \bar{\lambda} \equiv \frac{\lambda}{\pi^2}$

•magnons: small fluctuations near $\vec{n} = (0,0,1)$

$n_1 + in_2 = ae^{iwt+ikx}, \qquad w \sim \bar{\lambda} k^2 \ ,$

small amplitude, delocalized

•solitons: finite $E - J \sim \frac{\lambda}{J_2}, \ \ J_2$=finite

analogs of Bethe bound states in discrete model localised “pulse” soliton (Tjon, Wright 77; Fogedby 80)

$$n_3 = \cos\psi = 1 - \frac{2A}{\cosh^2[q(x-vt)]}, \qquad n_1 + i n_2 = \sin\psi\, e^{i\varphi}$$

$$\varphi = wt + b(x-vt) + \arctan(c \tanh[q(x-vt)])$$

parameters w and v: angular momentum J_2 and momentum p

$A = 1 - \frac{v^2}{\bar{\lambda} w}, \quad q = \sqrt{\frac{wA}{\bar{\lambda}}}, \quad c = \frac{\bar{\lambda} q}{v}, \quad b = \frac{\bar{\lambda}}{v}$

center at $x_0 = 0$, width q^{-1}, $\vec{n}_{|x| \to \infty} \ \longrightarrow \ (0, 0, 1)$

dispersion relation:

$$E = \frac{\bar{\lambda}}{2J_2} \sin^2 \frac{p}{2}$$

$p = \int dx \frac{n_1 n_2' - n_2 n_1'}{1+n_3}, \qquad J_2 = \int dx (n_3 - 1)$

$v \leq v_{max} = \sqrt{\bar{\lambda} w}$: $v \longrightarrow v_{max}$ is small amplitude/large width limit: $J_2 \longrightarrow 0$, $\frac{E}{J_2} \longrightarrow \omega$, $\frac{p}{J_2} \longrightarrow k$

soliton reduces to magnon with $\omega \sim k^2$

soliton: non-topological, continuously deformed into vac.

Scattering of solitons and magnons (Takhatjan 77)

Semiclassical quantization (Fogedby 80, Jevicki, Papanicolaou 79): $J_2 = 1$ quantum magnon with $w \sim k^2$ and $J_2 = 1, 2, ...$ quantum soliton with $E = \frac{\bar{\lambda}}{2J_2} \sin^2 \frac{p}{2}$

$J_2 = 1$ magnon and $J_2 = 1$ soliton are the same state in exact quantization: $w \sim k^2 \to w \sim \sin^2 \frac{k}{2}$

full quantum dispersion relation is reproduced by semiclassical quantization of soliton (exact due to integrability)

[analogy with sine-Gordon: magnon – basic excitation, soliton – breather (doublet); lowest state in doublet mass spectrum is same as basic "meson" (Dashen et al 77); cf. massive Thirring to XYZ model (Luther)]

Correspondence with discrete quantum $XXX_{1/2}$ ($J = \infty$): quantum magnon – elementary ($J_2 = 1$) magnon; quantum soliton – Bethe bound states of ($J_2 > 1$) magnons

Lessons for comparison with string theory

[cf. "giant magnon" (Hofman, Maldacena 06)]

Generalization to all orders in λ using BDS

• $E - J = 0$: ferromagnetic vacuum –point-like string

• $E - J = J_2\sqrt{1 + \frac{\lambda}{J^2}n^2}$: $J_1 \gg J_2 \sim 1,$ BMN magnons

– "short" fast strings with c.o.m. along S^5 geodesic

• $E - J = \frac{\lambda}{J}b_1 + \frac{\lambda^2}{J^3}b_2 + ...,$

"thermodynamic" limit: $J_2 \sim J \gg 1, \quad b_i = b_i(\frac{J_2}{J})$

– long fast strings (Frolov, AT 03)

• $E - J = c(J_2, \lambda)$: $J \gg J_2 > 1$

bound states of J_2 magnons – generalized "Bethe strings"

– limits ($J \to \infty$) of rotating strings with folds/spikes

• $E = f(\lambda)J$: anti-ferromagnetic state (+ spinons)

generalization of Hulten state using BDS ansatz (Rej, Serban, Staudacher 05; Zarembo 05):

$$f(\lambda) = 1 + \frac{\sqrt{\lambda}}{\pi}\int_0^\infty \frac{dk}{k} \frac{\mathrm{J}_0(\frac{\sqrt{\lambda}\,k}{2\pi})\mathrm{J}_1(\frac{\sqrt{\lambda}\,k}{2\pi})}{\mathrm{e}^{\,k}+1}$$

$$f_{\lambda\ll1} = 1 + 4\ln 2\,\frac{\lambda}{16\pi^2} - 9\zeta(3)(\frac{\lambda}{16\pi^2})^2 + \ldots$$

$$f_{\lambda\gg1} = \frac{\sqrt{\lambda}}{\pi^2} + \ldots$$

– long slowly-rotating circular string with $J_1 = J_2$:

$$E = \sqrt{J^2 + \lambda m^2}, \quad m = J \ , \quad E \to \sqrt{\lambda}\, J$$

(Roiban, Tirziu, AT 06)

Gauge vs string theory: different limits:

perturbative semiclassical string side: $\lambda \gg 1, J \gg 1,$

with $\frac{J}{\sqrt{\lambda}}$=fixed

perturbative gauge side: $\lambda \ll 1,$ then $J \gg 1$

still, in some cases few leading coefficients match exactly

(for BMN, fast strings, $J_1 = \infty$ strings): susy protection

general pattern: strong-weak coupling interpolation

Low-energy states: fast 2-spin strings

perturbative string: classical + quantum $\alpha' \sim \frac{1}{\sqrt{\lambda}}$

large λ, large J with fixed $\tilde{\lambda} \equiv \frac{\lambda}{J^2}$, then expand in $\tilde{\lambda}$

perturbative SYM: first small λ, then expand in large J

get same structure and same coefficients at first two orders

(Frolov, AT 03; Beisert,Minahan,Staudacher,Zarembo 03; Serban, Staudacher 03)

interpolating function of λ from "3-loop" order:

quantum string expansion near fast strings contains "non-analytic" terms with explicit factors of $\sqrt{\lambda}$

(Beisert, AT 05; Schafer-Nameki, Zamaklar 05))

$$E = J\Big[1 + \tilde{\lambda}(a_0 + \frac{a_1}{J} + ...) + \tilde{\lambda}^2(b_0 + \frac{b_1}{J} + ...) + \tilde{\lambda}^3(f(\lambda) + ...) + ...\Big], \qquad \tilde{\lambda} \equiv \frac{\lambda}{J^2}$$

interpolating function:

$$f_{\lambda \gg 1} = c_0 + \frac{c_1}{\sqrt{\lambda}} + ... , \qquad f_{\lambda \ll 1} = d_1 + d_2\lambda + ... ,$$

but $\qquad c_0 \neq d_1$

Effective field theory approach:

two "microscopical" theories – spin chain and superstring –

lower part of the spectrum approximated by low-energy 2d effective actions: slow modes at large J

lead to non-relativistic "Landau-Lifshitz" 2d action (Kruczenski, 2003; Kruczenski, Ryzhov, AT, 2004)

$\lambda \gg 1$ to $\lambda \ll 1$ interpolation between "string" and "gauge" effective actions and corresponding "spin chains"

Coherent-state action for low-energy excitations of spin chain (determined by H = dilatation operator)

and "fast-string" limit of string action

$\vec{n}$ – transverse position of string in S^3

or spin coherent state $U^\dagger \vec{\sigma} U = \vec{n}, \ \vec{n}^2 = 1$

LL action from string theory

(i) isolate "fast" coordinate α whose momentum p_α is large

(ii) gauge-fixe $t = \tau$ and $p_\alpha = J$ (or $\tilde{\alpha} = J\sigma$ where $\tilde{\alpha}$ is "T-dual" to α)

(iii) expand action in derivatives of "slow" coordinates, or in $\sqrt{\tilde{\lambda}} = \frac{1}{\mathcal{J}}$.

part of $AdS_5 \times S^5$ metric

$$ds^2 = -dt^2 + d\mathrm{X}_i d\mathrm{X}_i^*, \quad \mathrm{X}_i \mathrm{X}_i^* = 1$$

$$\mathrm{X}_1 = X_1 + iX_2 = U_1 e^{i\alpha} ,$$

$$\mathrm{X}_2 = X_3 + iX_4 = U_2 e^{i\alpha} , \quad U_a U_a^* = 1 ,$$

$$d\mathrm{X}_a d\mathrm{X}_a^* = (d\alpha + C)^2 + DU_a DU_a^*,$$

$$C = -iU_a^* dU_a, \quad DU_a = dU_a - iCU_a.$$

Introduce $\vec{n} = U^\dagger \vec{\sigma} U, \quad U = (U_1, U_2)$

$$d\mathrm{X}_a d\mathrm{X}_a^* = (D\alpha)^2 + \frac{1}{4}(d\vec{n})^2 , \qquad D\alpha = d\alpha + C(n)$$

key assumption: t evolution of U_a is slow

$$L = -iU_a^* \partial_t U_a - \frac{1}{2}\tilde{\lambda}|D_\sigma U_a|^2 + O(\tilde{\lambda}^2)$$

CP^1 LL action in terms of $\vec{n}$

LL action beyond leading order

effective actions from gauge-theory spin chain and string theory: $S = \int dt \int_0^J dx\, L \,, \qquad \bar\lambda = \frac{\lambda}{(2\pi)^2} \,, \quad x = J\sigma$

$$L = \vec{C}(n)\cdot\partial_t\vec{n} - \frac{1}{4}\vec{n}\,(\sqrt{1-\bar\lambda\partial_x^2}-1)\,\vec{n} - \frac{3\bar\lambda^2}{128}(\partial_x\vec{n})^4$$

$$+\frac{\bar\lambda^3}{64}\left[\frac{7}{4}\,(\partial_x\vec{n})^2(\partial_x^2\vec{n})^2 \;-\; \mathrm{b}(\lambda)\,(\partial_x\vec{n}\partial_x^2\vec{n})^2 \;-\; \mathrm{c}(\lambda)\,(\partial_x\vec{n})^6\right]+\ldots$$

quadratic part is exact: reproduces the BMN dispersion relation for small ("magnon") fluctuations near $\vec{n} = (0,0,1)$

$$\partial_t^2 - \partial_x^2 + m^2 \rightarrow (i\partial_t - \sqrt{m^2-\partial_x^2})(-i\partial_t - \sqrt{m^2-\partial_x^2})$$

and $\sqrt{1+4\bar\lambda\sin^2\frac{p}{2}} \;\rightarrow\; \sqrt{1+\bar\lambda p^2}, \quad p = \frac{2\pi n}{J} \rightarrow 0$

Orders $\bar\lambda$ and $\bar\lambda^2$: direct agreement

"3-loop" coefficients are interpolating functions:

$$\lambda \gg 1 \quad : \qquad \mathrm{b} = -\frac{25}{2} + O(\frac{1}{\sqrt{\lambda}})\,, \qquad \mathrm{c} = \frac{13}{16} + O(\frac{1}{\sqrt{\lambda}})$$

$$\lambda \ll 1 \quad : \qquad \mathrm{b} = -\frac{23}{2} + O(\lambda)\,, \qquad \mathrm{c} = \frac{12}{16} + O(\lambda)$$

implied by non-analytic terms in 1-loop string correction:
$J\tilde\lambda^3\frac{1}{\sqrt{\lambda}} = \frac{\lambda^{5/2}}{J^5}$ (Beisert, AT; Schafer-Nameki, Zamaklar 05)

“Intermediate” part of spectrum: $J \to \infty, \; J_2 < \infty$

Remarkably, Bethe bound states admit direct generalization to all-order BDS ansatz

- poles in the BDS S-matrix (Dorey 06)

$$E - J_1 = \sqrt{J_2^2 + \frac{\lambda}{\pi^2} \sin^2 \frac{p}{2}}$$

- Bethe string solutions of BDS BA (Minahan, Tirziu, AT 06)

same distribution of u_j with $u_j = \frac{1}{2} \cot \frac{p_j}{2} \sqrt{1 + \frac{\lambda}{\pi^2} \sin^2 \frac{p_j}{2}}$

$p = \sum_1^{J_2} p_j = 2\, \mathrm{Im} \left[\operatorname{arccosh} \frac{u_0 + \frac{i}{2} J_2}{\sqrt{\lambda}/2\pi} \right]$

Generalization: bound state of n magnons with $\frac{J_2}{n}$ and $\frac{p}{n}$

$$E - J_1 = \sqrt{J_2^2 + \frac{\lambda}{\pi^2} n^2 \sin^2 \frac{p}{2n}} = n \sqrt{(\frac{J_2}{n})^2 + \frac{\lambda}{\pi^2} \sin^2 \frac{p}{2n}}$$

Same in AFS case; assuming dressing factor $e^{i\theta}$ has no poles or zeroes should be true in general

String interpretation?

Large J limit of semiclassical closed strings as "Bethe strings":

semiclassical strings: $\lambda \gg 1, \ \ \mathcal{E} = \frac{E}{\sqrt{\lambda}}, \ \mathcal{J}_i = \frac{J_i}{\sqrt{\lambda}}$ fixed

special limit: $\mathcal{J}_1 \to \infty, \ \mathcal{J}_2$=fixed

general pattern: $E \to \infty, \ J_1 \to \infty, \ \ E - J_1$ =finite:

$$E - J_1 = \sqrt{J_2^2 + c\lambda} \, , \qquad c = \text{const}$$

"Infinitely long/heavy solitonic strings": have special properties (BPS-like non-renormalization of classical energy,...)

(Hofman, Maldacena; Dorey; Chen, Dorey, Okamura; Arutyunov, Frolov, Zamaklar; Minahan, Tirziu, AT; Spardlin, Volovich; Kruczenski, Russo, AT)

Examples:

Limit of folded string with two spins on S^3

$$ds^2 = -dt^2 + d\theta^2 + \cos^2\theta \, d\varphi_1^2 + \sin^2\theta \, d\varphi_2^2$$

$$t = \kappa\tau \ , \quad \theta = \theta(\sigma) \ , \quad \varphi_1 = w_1\tau \ , \quad \varphi_2 = w_2\tau \ ,$$

(Frolov, AT 03)

in the limit $\mathcal{J}_1 \to \infty, \ \ \mathcal{J}_2 =$fixed:

$w_1 = \kappa, \ \ w_2 = w\kappa, \ \ \kappa \to \infty$

string maximally stretched $\theta_{max} = \frac{\pi}{2}$: angular momentum J_1 around c.o.m. is maximal ($J_1 = \infty$) (Dorey 06)

$$E - J_1 = \sqrt{J_2^2 + \frac{4\lambda}{\pi^2}}$$

Special case of limit of rotating string with spikes

(Ryang 05; Minahan, Tirziu, AT 06)

$$E - J_1 = \sqrt{J_2^2 + \frac{\lambda}{\pi^2} n^2 \sin^2 \frac{p}{2n}}$$

closed string solution: $p = 2\pi m$, n=number of spikes, m=winding ($\varphi_1 = \omega_1 \tau + m\sigma$)

Interpretation: built out of n "giant magnons" (each with $\frac{J_2}{n}$)

Classical theory: $J_2 \gg 1$; in quantum theory J_2 can be =1:

$E - J_1 = \sqrt{1 + \frac{\lambda}{\pi^2} \sin^2 \frac{p}{2}}$

cf. soliton in the LL model on a line: $J_2 \neq 1$;

quantum $J_2 = 1$ soliton is same as quantum magnon giant magnon with $J_2 \neq 0$ (Chen, Dorey, Okamura) reduces to soliton of LL when expanded in $\frac{1}{\mathcal{J}_2} = \frac{\lambda}{J_2^2}$

$J_2 = 0$ case should be understood as formal limit of $J_2 \neq 0$

Vanishing of 1-loop string correction to energy

(Minahan, Tirziu, AT):

suggests non-renormalization of the classical energy formula in semiclassical expansion with $\mathcal{J}_i \equiv \frac{J_i}{\sqrt{\lambda}}$ and p fixed:

$$E - J_1 = \sqrt{\lambda}\sqrt{\mathcal{J}_2^2 + \frac{n^2}{\pi^2}\sin^2\frac{p}{2n}} + 0 + O(\frac{1}{\sqrt{\lambda}})$$

Limit of circular 2-spin solution

$$t = \kappa\tau, \quad \theta = \theta_0 = \text{const}, \ \ \varphi_i = w_i\tau + m_i\sigma \ ,$$

(Arutyunov, Russo, AT 03)

$$m_1 J_1 + m_2 J_2 = 0$$

if $J_1 \gg J_2$ then $m_2 \gg m_1$

limit: $w_1 = \kappa \to \infty, \ m_2 \to \infty$

$$E - J_1 = \sqrt{J_2^2 + \lambda m_1^2}$$

infinitely wound string; special case of

$$E - J_1 = \sqrt{J_2^2 + \frac{n^2}{\pi^2} \sin^2 \frac{p}{2n}}$$

when $p = 2\pi m_1$ and $n \to \infty$ (infinite "Bethe string")

Vanishing of 1-loop string correction to classical energy:

non-trivial cancellation between 2d bosons and fermions

(hidden 2d susy ?)

$$E_1 = \frac{1}{2} \int_{-\infty}^{\infty} dp \Big[6\sqrt{p^2 + 1} + \sqrt{(p+\gamma)^2 + 1} + \sqrt{(p-\gamma)^2 + 1} - 4\sqrt{\left(p + \tfrac{1}{2}\gamma\right)^2 + 1} - 4\sqrt{\left(p - \tfrac{1}{2}\gamma\right)^2 + 1} \Big] = 0, \qquad \gamma \equiv (\mathcal{J}_2)^{-1}$$

Generalizations:

limit of pulsating solution on S^3 (Minahan et al)

$$E - J = N\sqrt{1 + q^2\lambda}\,, \qquad q = \frac{m}{J} =$$

scattering states of giant magnons on S^3 (Spradlin, Volovich)

2-magnon superposition on S^5 (in $SU(3)$ sector) (Kruczenski, Russo, AT)

Some conclusions

- Correspondence between gauge and string spectra near and far from BPS limit
- Presence of non-trivial interpolation functions in Bethe ansatz phase, string energies, effective LL action
- Special large J limit: simplicity, non-renormalizability, relation to S-matrix
- Compute soliton scattering? Additional constraints on S-matrix?