AEI Potsdam
July 2006

N. Dorey
Cambridge
DAMTP

Magnon Boundstates and the AdS/CFT Correspondence

hep-th/0604175
+ H.Y. Chen, ND, K. Okamura,
hep-th/0605155
and work in progress....

AdS/CFT equates:

- Spectrum of operator dimensions in planar $\mathcal{N}=4$ SUSY Yang-Mills
- Spectrum of free strings on $AdS_5 \times S^5$

dual theories weakly coupled in different limits

i) $\lambda \ll 1 \Rightarrow \mathcal{N}=4$ SYM weakly coupled

ii) $\lambda \gg 1 \Rightarrow$ string σ-model weakly coupled

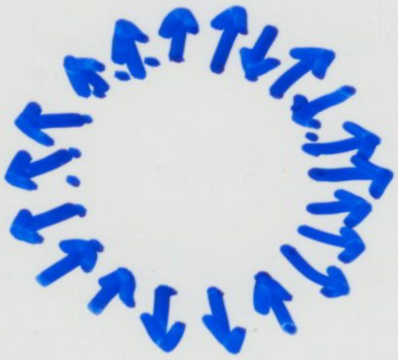
't Hooft coupling: $\lambda = g^2 N$

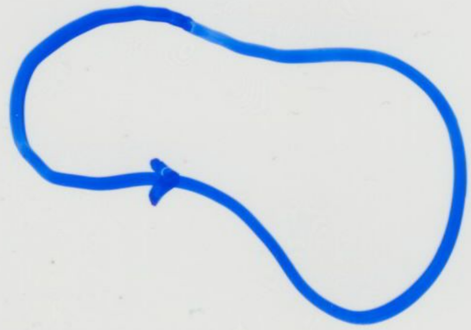
Gauge Theory

Minahan + Zarembo

String Theory

Bena, Polchinski + Roiban





Periodic BC,

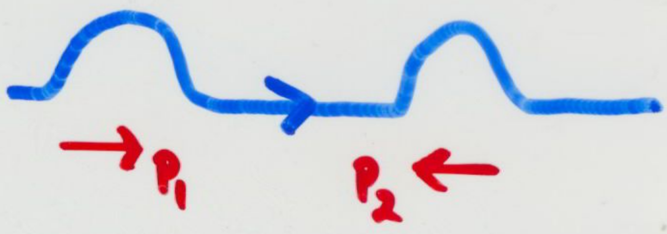
Integrability $\overset{?}{\Longrightarrow}$ Bethe Ansatz

- $\underline{J \to \infty \text{ limit}} \Rightarrow$ long chain/string

Staudacher
Beisert
Hofman + Maldacena

$\rightarrow p_1 \qquad\qquad p_2 \leftarrow$

$\ldots\uparrow\uparrow\downarrow\uparrow\uparrow\ldots\ \uparrow\downarrow\uparrow\uparrow\ldots$



Vacuum BC,

Integrability $\Rightarrow$ Factorised scattering

S-matrix | Spectrum

singularities $\leftrightarrow$ boundstates, anomalous thresholds ....

planar $\mathcal{N}=4$ SYM, $J \to \infty$ $\forall \lambda$

- Spectrum, Beisert
  - magnons
  - boundstates

exact dispersion relation,

$$\Delta - J = \sqrt{Q^2 + \frac{\lambda}{\pi^2} \sin^2\left(\frac{p}{2}\right)}$$

Q = # of constituents

Beisert, Dippel + Staudacher

Beisert

- S-matrix,

almost determined ....

Beisert

SU(2) sector at one-loop

Minahan + Zarembo

$\mathcal{N}=4$ operators of form,

$$\mathrm{Tr}_N\left[X^{J_1} Y^{J_2}\right]$$

$$\Updownarrow$$

configurations of Heisenberg $XXX_{1/2}$ spin chain,

length; $L = J_1 + J_2$

impurity #; $M = J_2$

Heisenberg Hamiltonian

$$\hat{D} = L\,\mathbb{1} + \frac{\lambda}{8\pi^2}\hat{H} + O(\lambda^2)$$

$\mathcal{N}=4$ dilatation operator

eigenvalues,

$$\Delta = L + \frac{\lambda}{8\pi^2}E + O(\lambda^2)$$

infinite chain: $L \to \infty$, M fixed

<u>$M=0$</u> ferromagnetic ground state

$\ldots\ldots\uparrow\uparrow\uparrow\uparrow\uparrow\ldots$

<u>$M=1$</u> one impurity

$|\ell\rangle = \ldots\ldots\uparrow\uparrow\downarrow\uparrow\uparrow\ldots\ldots$

ℓ'th site

Magnon:

$$|p\rangle = \sum_{\ell} e^{ip\ell}|\ell\rangle \qquad \psi_p(\ell) = e^{ip\ell}$$

$\ldots\ldots\uparrow\uparrow\downarrow\uparrow\uparrow\ldots$

$\longrightarrow p$

Dispersion Relation:

$$\varepsilon(p) = 4\sin^2(p/2)$$

$\underline{M=2}$ two magnon state,

$$\ldots\uparrow\uparrow\downarrow\uparrow\uparrow\ldots\ldots\ldots\uparrow\uparrow\downarrow\uparrow\uparrow\ldots$$

$$\xrightarrow{\quad} p_1 \qquad\qquad p_2 \xleftarrow{\quad}$$

wavefunction,

$$\psi_{p_1,p_2}(\ell_1,\ell_2) = e^{i\ell_1 p_1 + i\ell_2 p_2} \qquad \ell_1 < \ell_2$$

$$= S(p_1,p_2)\, e^{i\ell_1 p_1 + i\ell_2 p_2} \qquad \ell_1 > \ell_2$$

is energy eigenstate for,

$$S(p_1,p_2) = \frac{u(p_1) - u(p_2) + i}{u(p_1) - u(p_2) - i}$$

S-matrix

with,

$$u(p) = \tfrac{1}{2}\cot(p/2)$$

S-matrix has pole at complex momentum,

$$\varphi(p_1) - \varphi(p_2) = i \quad \cdot \circledast$$

$\varphi(p) = \frac{1}{2}\cot\left(\frac{p}{2}\right)$

set $p_1 = \frac{P}{2} + iv$

$p_2 = \frac{P}{2} - iv$

$\circledast \Rightarrow e^{v} = \cos(P/2)$

normalised two-magnon wave function,

$$\Psi_{p_1,p_2}(\ell_1,\ell_2) = e^{+v|\ell_1-\ell_2|} \times e^{iP(\ell_1+\ell_2)/2}$$

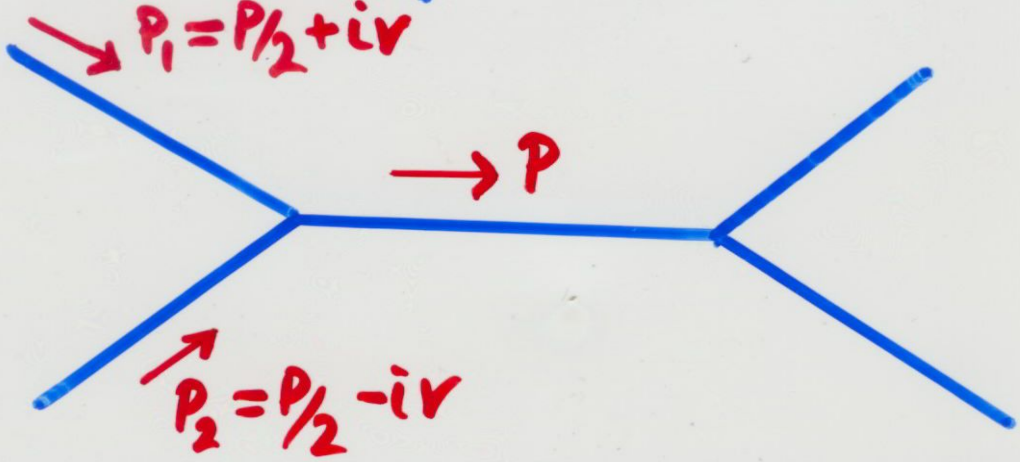
$$= \left[\cos(P/2)\right]^{|\ell_1-\ell_2|} \times e^{iP(\ell_1+\ell_2)/2}$$

decays for $|\ell_1-\ell_2| \to \infty$

COM wavefunction

$\Rightarrow$ normalisable boundstate with $J_2 = 2$

boundstate formed in "s-channel",



boundstate dispersion relation,

$$\begin{aligned}\mathcal{E}_2(p) &= \mathcal{E}(p/2 + iv) + \mathcal{E}(p/2 - iv) \\ &= 2\sin^2(p/2) \\ &= \mathcal{E}(p)/2\end{aligned}$$

$\mathcal{E}(p) = 4\sin^2(p/2)$

note that,

$$\mathcal{E}_2(p) \leqslant \mathcal{E}(q) + \mathcal{E}(p-q) \quad \forall p, q \in \mathbb{R}$$

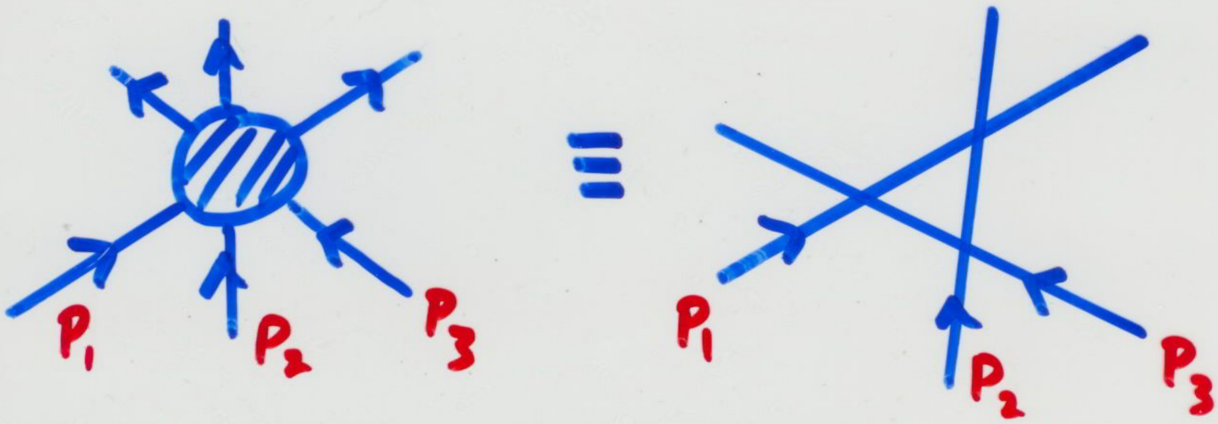
$\Rightarrow$ stability

M > 2

integrability $\Rightarrow$ factorized scattering

Polyakov
Faddeev
Zamolodchikov

3-body S-matrix,

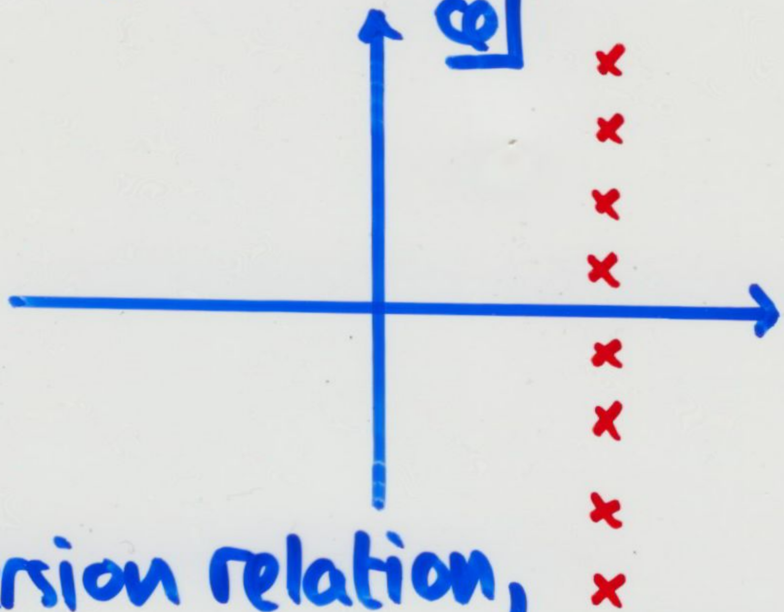


$$S_{123} = S_{23}\, S_{13}\, S_{12}$$

3-magnon boundstate corresponds to pole at,

$$u_1 - u_2 = u_2 - u_3 = i$$

Q-magnon boundstates correspond to "Bethe strings"



$\varphi_j - \varphi_{j+1} = i$

$j = 1, \ldots, Q-1$

dispersion relation,

$$\underline{\underline{\varepsilon_Q = 4/Q \, \sin^2(p/2)}}$$

$Q = 1, 2, 3, \ldots$

correspond to operators in $\mathcal{N}=4$ SYM of form,

$$\hat{O} \sim \mathrm{Tr}\left[\ldots\ldots XX\,Y^Q XX\ldots\right]$$

Q impurities bound together

full spectrum in $L \to \infty$ limit
M fixed

$\equiv$ Free multi-particle Fock space

$$\hat{O} \sim \mathrm{Tr}[\ldots\ldots XXYX\ldots XXYXX\ldots XYX\ldots]$$

Q_1, Q_2, Q_3

$\overrightarrow{p_1}$ $\overleftarrow{p_2}$ $\overrightarrow{p_3}$

$$\Delta = L + \frac{\lambda}{8\pi^2} \sum_j \frac{4}{Q_j} \sin^2\left(\frac{p_j}{2}\right) + \ldots$$

What happens,

- Beyond SU(2) sector ?
- Beyond one-loop ?

Full Planar $\mathcal{N}=4$ Theory.

$J_1 \to \infty \quad \Delta - J_1, \lambda$ held fixed   Beisert

- ferromagnetic groundstate,

$$\ldots. XXXXXXX \ldots.$$

has unbroken SUSY,

$$SU(2|2) \times SU(2|2)$$

linearly realised on individual impurities,

$$\ldots. XXX \overset{\longrightarrow p}{I} XXX \ldots.$$

$$I \in \{ Y, Z, \mathcal{D}_\mu, \lambda_\alpha, \bar{\lambda}_{\dot{\alpha}} \}$$

with non-trivial central extension

$$\{ Q, Q \} = \mathbb{P} \sim (e^{ip} - 1)$$

$\uparrow$ SUSY generators

- impurities form <u>short rep.</u>

$\underline{16} = (\square, \square)$ of $SU(2|2)^2$

with <u>exact</u> dispersion relation

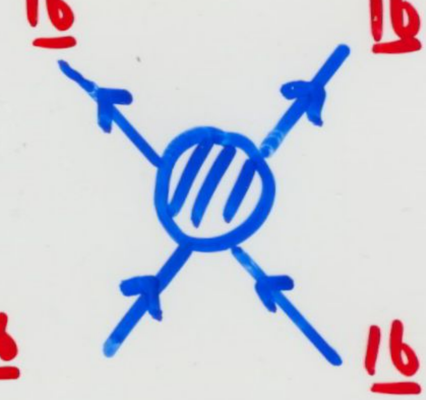
$$\Delta - J_1 = \sqrt{1 + \frac{\lambda}{\pi^2} \sin^2(p/2)}$$

Beisert, Dippel, Staudacher ....

<u>Magnon S-matrix</u>

$$\hat{S}_{IJ}(p_1, p_2, \lambda)$$

$\nwarrow$ 256×256



determined up to an overall phase by SUSY Beisert

SU(2) sector

$$= \sigma(p_1, p_2; \lambda) \begin{pmatrix} S_{BDS} & \cdots \\ \vdots & \end{pmatrix}$$

dressing factor

$$S_{BDS} = \frac{\varphi(p_1) - \varphi(p_2) + i}{\varphi(p_1) - \varphi(p_2) - i}$$

$$\varphi(p) = \tfrac{1}{2}\cot(p/2)\sqrt{1 + \lambda/\pi^2 \sin^2(p/2)}$$

- pole at $\varphi(p_1) - \varphi(p_2) = i$ persists

same calculation $\Rightarrow$
2-magnon boundstate with,

$$\Delta - J_1 = \sqrt{4 + \lambda/\pi^2 \sin^2(p/2)}$$

also find Q-magnon boundstate with exact dispersion relation,

$$\Delta - J_1 = \sqrt{Q^2 + \frac{\lambda}{\pi^2}\sin^2(p/2)}$$

<u>Explanation</u>

$Q = 1, 2, \ldots$

These states have $J_2 = Q$

SUSY algebra

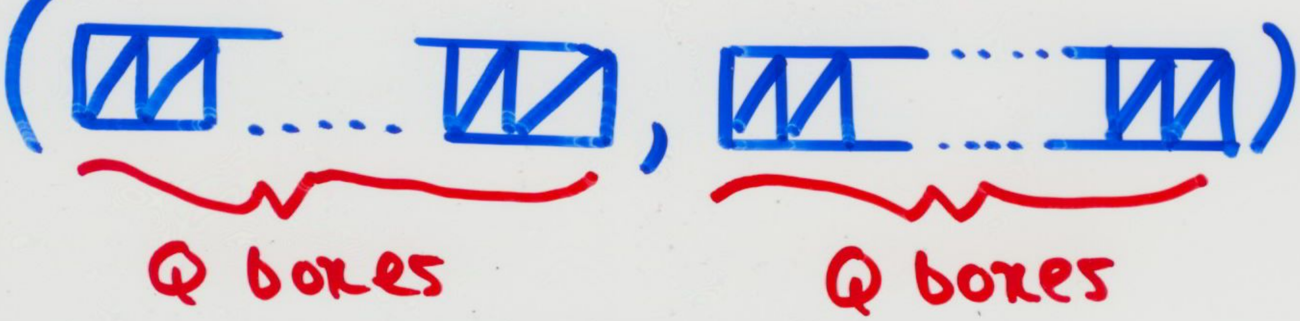
$$\{Q, Q\} \sim J_2 + P$$

↑ Beisert central charge

allows short-multiplets with protected dispersion relation,

$$\Delta - J_1 = \sqrt{J_2^2 + \frac{\lambda}{\pi^2}\sin^2(p/2)}$$

Q-magnon boundstates form short reps, Chen, ND, Okamura to appear



of $SU(2|2) \times SU(2|2)$

- BPS property $\Rightarrow$ states should persist $\forall \lambda$

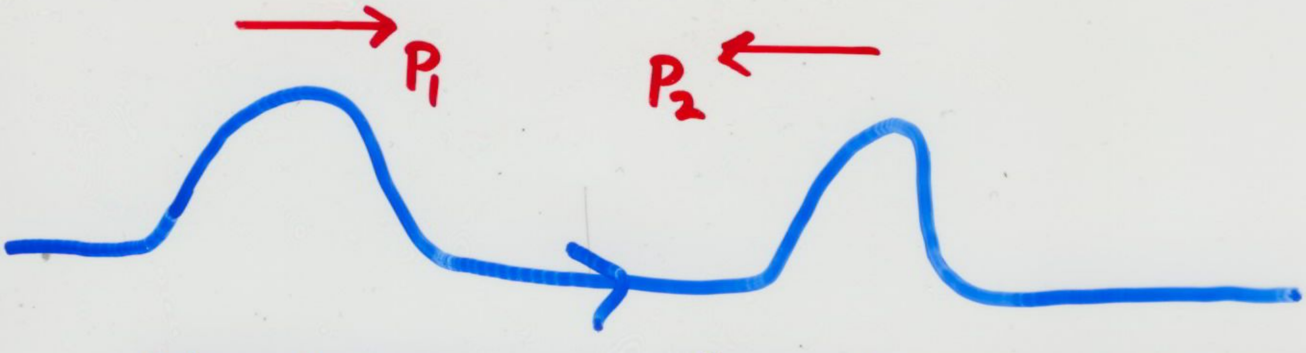
in particular take $Q \sim \sqrt{\lambda} \gg 1$

$$\Delta - J_1 = \sqrt{Q^2 + \frac{\lambda}{\pi^2} \sin^2\left(\frac{p}{2}\right)} \sim \sqrt{\lambda}$$

... should be visible in semiclassical string theory on $AdS_5 \times S^5$

String Theory.

$J_1 \to \infty$ limit yields infinitely long string, Hofman + Maldacena



magnons $\equiv$ classical solitons

One-charge states $J_1 \to \infty$

S^2 σ-model $\xrightarrow{\text{reduction}}$ sine-Gordon

Two-charge states $J_1 \to \infty$, J_2 fixed

S^3 σ-model $\xrightarrow{\text{reduction}}$ complex sine-Gordon

Pohlmeyer

Complex sine-Gordon Eqn

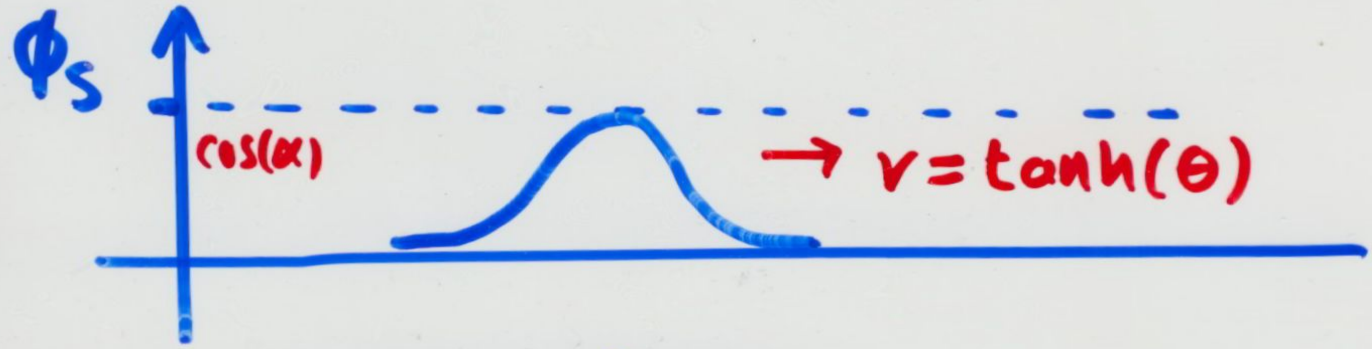
integrable PDE for $\psi(\sigma,\tau) \in \mathbb{C}$,

$$\partial_+\partial_-\psi + \psi^* \frac{\partial_+\psi\,\partial_-\psi}{1-|\psi|^2} + \psi(1-|\psi|^2) = 0$$

Getmanov
Lund + Regge
......

Soliton solution,

$$\psi(\sigma,\tau) = e^{i\sin(\alpha)\left[\frac{\tau - v\sigma}{\sqrt{1-v^2}}\right]}\,\phi_s\left[\frac{\sigma - v\tau}{\sqrt{1-v^2}}\right]$$



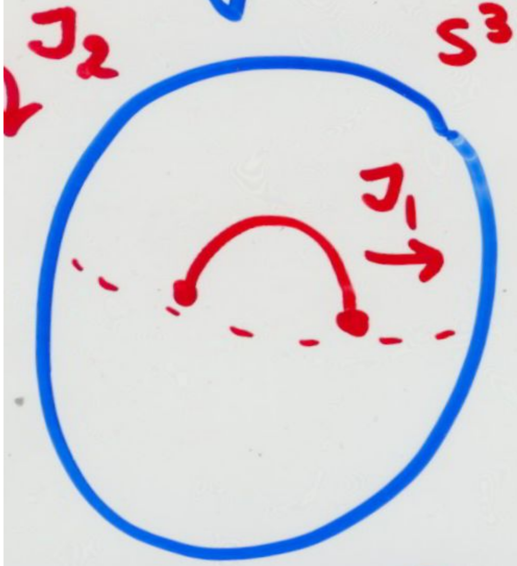
two parameters,

θ ~ rapidity

α ~ rotation parameter

reconstruct corresponding string motion,

"Dyonic giant magnon"



Chen, ND, Okamura
Arutyonov, Frolov, Zamaklar
Minahan, Tirziu, Tseytlin
Spradlin, Volovich

dictionary,

$$J_2 = \frac{\sqrt{\lambda}}{\pi} \frac{s(\alpha)c(\alpha)}{c^2(\alpha) + sh^2(\theta)}$$

$$\Delta - J_1 = \frac{\sqrt{\lambda}}{\pi} \frac{sh(\theta)ch(\theta)}{c^2(\alpha) + sh^2(\theta)}$$

$$\cot(p/2) = 2\,sh(\theta)/c(\alpha)$$

$$\Rightarrow \Delta - J_1 = \sqrt{J_2^2 + \frac{\lambda}{\pi^2}\sin^2(p/2)}$$

Further progress

Chen, ND, Okamura to appear

- CSG solitons undergo factorised scattering with known time delay,

$$\Delta T_{COM} = \frac{2}{sh(\beta_1)\,c(\alpha_1)} \log \left| \frac{sh(\Delta\theta + i\Delta\alpha)}{ch(\Delta\theta + i\bar{\alpha})} \right|$$

$\Rightarrow$ semiclassical worldsheet S-matrix via

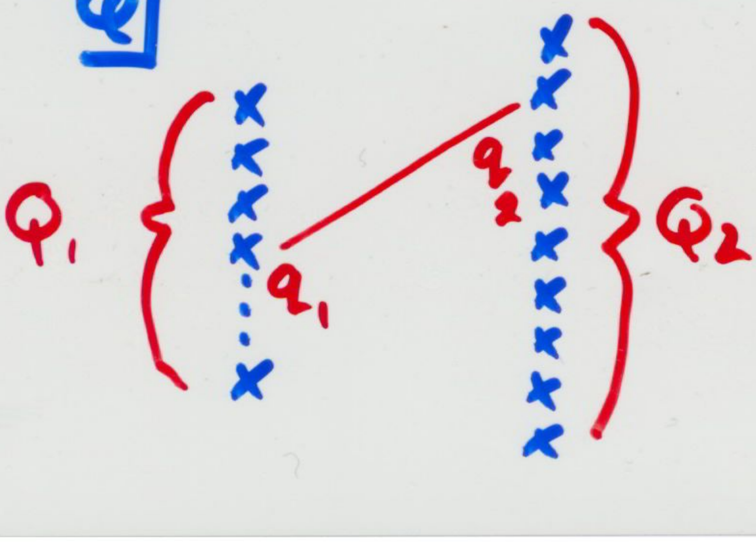
$$S = e^{i\delta}, \quad \partial\delta/\partial E = \Delta T$$

Jackiw + Woo

- **Gauge Theory Calculation**

scattering of two "Bethe strings"

eg



$$S = \prod_{q_1, q_2} S^{(1)}(q_1, q_2)$$

use,

$$S^{(1)} = \sigma_{AFS} \times \hat{S}_{BDS}$$

$\Rightarrow$ exact agreement with string theory for $\sqrt{\lambda} \gg 1$

Open Questions

- other non-BPS boundstates
  SG (CSG) breathers
- constraints on $\hat{S}$-matrix from singularities
- Bootstrap?